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Prediction Procedures Used in Satellite Catalog Maintenance

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Principles of prediction procedures used in the maintenance of the satellite catalog are described. These algorithms allow propagation of satellite motion from the time of launch up to reentering and have very small computer-time requirements. The techniques, element sets, precision, and computer time are given. The issues of conversion of the element sets (catalog element sets) to NORAD two-line elements are treated.

Nomenclature

a	=	semimajor axis
c	=	$\cos(i/2)$
E	=	eccentric anomaly
e	=	eccentricity
f	=	true anomaly
h	=	$e \sin \omega$
i	=	inclination
k	=	$e \cos \omega$
L	=	$\sqrt{\mu a}$
l	=	mean anomaly
n_e	=	Earth rotation rate
R_e	=	Earth radius
S	=	ballistic coefficient
S_{star}	=	star time (Greenwich hour angle)
s	=	$\sin(i/2)$
T_Ω	=	draconic (nodal) period
t	=	element-set epoch
t_Ω	=	time of equator crossing
α_{er}	=	Earth-oblateness coefficient
θ	=	$\cos i$
λ	=	$l + \omega$
μ	=	gravitational constant
Ω	=	longitude of ascending node
ω	=	argument of perigee

Introduction

ABOUT 10 million propagations for different orbital parameters are performed daily in the Space Surveillance Center (SCC). That is why the choice of the prediction technique providing the smallest computer time for the required accuracy level is a subject of special concern. Prediction algorithms developed by different authors vary with regard to the following major characteristics: 1) composition and degree of completeness in accounting for different perturbing factors, 2) technique of integration of the equations of motion, and 3) independent variable and element set.

Many different algorithms may be obtained, combining those factors in various ways (within possible limits). As a first example of analytic prediction algorithms, we consider the well-known theory of motion proposed by Gaposchkin [1] for geodetic satellites at altitudes of 700–4000 km. This theory obtains linear perturbations from the Earth, moon, and sun gravitational fields, tides, Earth precession, and nutation using time as the independent variable. Perturbations from the atmosphere and light pressure are accounted for as functions of eccentric anomaly, and true anomaly is used as the independent variable for second-order perturbations from Earth oblateness. The second example, closer to our subject, shows the prediction procedures used at the first stage of the Russian SSC development. The Russian element set (REL) [2] orbital parameters are used as the element set in these procedures, consisting of Kepler's elements i, Ω, e, and ω (or h and k instead of the latter two) at the time of passage over the ascending node t_Ω. The draconic period T_Ω is used instead of the semimajor axis, and instead of a ballistic coefficient, the decline of the period for a revolution ΔT is used.

For the Nth revolution, we have the following relationships:

$$T_{\Omega,N} = t_{\Omega,N} - t_{\Omega,N-1} \tag{1}$$

$$\mathbf{\Delta}\, T_{\Omega,N} + (\mathbf{\Delta} T_{\Omega,N})_{\text{grav}} = T_{\Omega,N} - T_{\Omega,N-1} \tag{2}$$

where $(\mathbf{\Delta} T_{\Omega,N})_{\text{grav}}$ accounts for the decline of the period due to gravitational perturbations. The independent variable is u, which is the osculating angular distance from the ascending node.

The merit of the REL data is the simplicity of their understanding. In essence, these elements are mean, because their determination for the beginning of the revolution $t_{\Omega,N}$ only takes into account the secular and long-periodic perturbations. But they are defined in such a way that for a certain time $t_{\Omega,N}$, they coincide with the osculating elements, and the user may forget the mean elements.

Prediction of the REL elements is divided into two stages. The first stage provides prediction for the entire number of revolutions, and the second stage carries out the prediction within the required revolution for the required value of the latitude argument u. Prediction for the required time t is accomplished by an iterative method on $u(t)$. The algorithm accounts for the zonal harmonics C_{20}, C_{30}, C_{40}, C_{50}, and C_{60} and atmospheric perturbations.

Various algorithms suggested by different authors were analyzed regarding the possibility of achieving higher calculation speed in the course of software development for the SSC modernization. The following decisions were the results of this analysis:

1) Time is chosen as the independent variable.
2) Mean Delaunay elements are chosen as the basic element set.
3) Brouwer's method is chosen for integration.

These choices increase the calculation speed with respect to the aforementioned first-stage algorithms, due to the following factors:

1) No iterations in $u(t)$ are needed for propagating to a given time.
2) No conversion of T_Ω to the semimajor axis is needed.
3) Perturbations from any geopotential harmonic and for any element are expressed in a general form via the Hamiltonian $\mathcal{H}$, providing possibilities for extensive use of effective recursive procedures.
4) Averaged equations are simplified due to the absence of additional components arising from conditions of coincidence of mean and osculating element sets for the beginning of the revolution.

A brief description of the prediction software tools package, developed on the basis of these considerations, is given subsequently. A paper describing both the employed techniques and the realized procedures would be too large. Thus, the material is divided into two parts: the basic paper and appendices. The basic paper

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briefly describes the techniques used in design of the procedures. The appendices provide a detailed description of the most frequently used propagators in the SSC.

An extensive set of prediction algorithms based on the Brouwer technique now exists. To define a reasonable domain of suggested algorithms within this set, the paper presents data regarding the achievable accuracy level and calculation speed for certain procedures of this package. The final section deals with the issues of the conversion of catalog element sets (CELs) to REL and to two-line element sets (TLEs) [3].

Element Sets

The initial data for prediction are element sets of the SSC satellite catalog. The elements will be further called CELs, which are averaged orbital elements $\{t, \lambda, L, \theta, \Omega, h, k, \text{and } S\}$, defined in the Nomenclature.

Algorithm Package

The prediction-algorithm package contains the following basic procedures: 1) analytical prediction algorithm (A), 2) analytical prediction algorithm with enhanced accuracy (AP), 3) numerical–analytical (combined) prediction algorithm with enhanced accuracy (NA), 4) numerical prediction algorithm (N), and 5) numerical prediction algorithm with enhanced accuracy (NP).

All prediction algorithms account for perturbations from the zonal harmonics C_{20}, C_{30}, C_{40}, C_{50}, and C_{60}; the second sectoral harmonic C_{22}; D_{22}; and atmospheric drag. The analytical prediction algorithm with enhanced accuracy additionally accounts for major perturbations from the zonal and tesseral harmonics up to the eighth order, acquired from Kaula's [4] formulas. The numerical prediction algorithm with enhanced accuracy additionally accounts for perturbations from all harmonics up to 14th order and lunar and solar perturbations.

The procedures possess special modes of ongoing calculations, providing effective motion predictions for sets of consequent times. For the ongoing calculations mode, the A, AP, and NA procedures remember the coefficients characterizing secular and long-periodic evolutions; the AP and NA procedures additionally remember tesseral harmonic perturbation amplitudes. In fact, only short-periodic perturbations from C_{20} are recalculated for propagating to each new time when the ongoing mode is used. This calculation mode is the basic one for processing observations by the least-squares method.

Analytical Prediction Algorithm

The algorithm uses the nonsingular elements CEL derived from the Delaunay elements:

$$\mathbf{w} = (L, G, H, l, g, h)$$

where

$$L = \sqrt{\mu a}, \qquad G = L\sqrt{1-e^2}, \qquad H = G\theta, \qquad \theta = \cos i$$
$$l = M, \qquad g = \omega, \qquad h = \Omega$$

The canonical equations of motion

$$\frac{d\mathbf{w}}{dt} = \mathcal{F}\frac{\partial \mathcal{H}(\mathbf{w})^T}{\partial \mathbf{w}}$$

where the

$$\mathcal{F} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

simplex matrix is transformed to secular [Eq. (3)] and long-periodic [Eq. (4)] equations using the Delaunay–Brouwer technique [5]:

$$\frac{d(l'', g'', h'')}{dt} = -\frac{\partial \mathcal{H}^{**}}{\partial(L'', G'', H'')}$$
$$G' = G'' + \frac{\partial \mathbf{S}^*}{\partial g'}$$
$$\Delta(l', g', h') = \frac{\partial \mathbf{S}^*}{\partial(L'', G'', H'')}$$

$$\mathcal{H}^{**} = \frac{\mu^2}{2L^2} + \mathcal{H}_B^{**} + \sum_{n=3}^{6} \frac{c_{n0} a_0^n \mu^{n+2}}{G^{2n-1} L^3} M_{n-1}^0(e) P_n^0(0) A_n^0(i) \qquad (3)$$

$$\mathbf{S}^* = \mathbf{S}_B^* - \frac{8}{3(1-5\theta^2)} \sum_{n=3}^{6} \frac{c_{n0} a_0^{n-2} \mu^{n-2}}{G^{2n-5} c_{20}} \sum_{k=1}^{n-1} \frac{M_{n-1}^k(e) P_n^k(0) A_n^k(i) \sin^k i}{2^k k! k} \qquad (4)$$

where $\mathcal{H}_B^{**}$ is the expression for the doubly averaged C_{20} perturbation function [5] of the second order of accuracy, $\mathbf{S}_B^*$ is the expression for the C_{20} generating function [5], $A_n^k(i)$ are the inclination Gooding [6] functions, $M_n^k(e)$ are the eccentricity functions [2], and $P_n^k(\cdot)$ are the associated Legendre functions. Here, single prime stands for the singly averaged variable (short-period perturbations are excluded), and double prime means a twice-averaged variable (the long-period perturbations are excluded as well).

For the A_n^k and M_n^k functions, specific efficient recursive formulas are developed. The details of all equations used in the A model are provided in Appendix A.

Analytical Prediction Algorithm with Enhanced Accuracy

For the A algorithm, prediction errors due to an incomplete account of geopotential perturbations are of the order of 1 km. Enhancement of the prediction accuracy to the value of 100 m is possible by transition to the 8×8 field.

Detailed analysis of the relationship between a procedure's efficiency (CPU time) and its accuracy revealed that a certain critical point exists beyond which a small increase in accuracy requires a great increase of CPU time. It seems to be reasonable to set the accuracy next to this critical point. It was revealed for the 8×8 field that long-periodic (daily) perturbations from tesseral harmonics and short-periodic (satellite period) perturbations for the second and third harmonics need to be calculated. The following limit for orbital eccentricity was introduced to attain a tolerable CPU time:

$$e < 0.05$$

The complex expression for the geopotential perturbing function in the format of Kaula [4] can be written as follows:

$$\mathcal{H} = \sum_{n=2}^{\infty} \sum_{q=1}^{n} \sum_{k=-n}^{n} \sum_{p=-\infty}^{\infty} \mathcal{H}_{nqkp} \qquad (5)$$

$$\mathcal{H}_{nqkp} = (-1)^{\frac{n-q}{2}} I_{nq} \left(\frac{\mu}{a}\right) \frac{a_0^n}{a^n} A_{nq}^k(i) X_p^{-(n+1),k}$$
$$\times \exp(\sqrt{(-1)}(pl + kg + q\Omega_{nq})) \qquad (6)$$

where μ is the gravitational constant, a_0 is the semimajor axis of the general Earth ellipsoid, $A_{nq}^k(i)$ is the inclination function [6], $X_p^{-(n+1),k}(e)$ is the eccentricity function, I_{nq} and λ_{nq} are the parameters of the Earth gravitational field, n_e is the angular velocity of Earth rotation, and $\Omega_{nq} = \Omega - n_e \Delta t - \lambda_{nq}$.

The expression for the generating function is derived from the equations

$$\mathbf{S} = \sum_n \sum_q \sum_k \sum_p \mathbf{S}_{nqkp}, \qquad \mathbf{S}_{nqkp} = \sqrt{-1}\frac{\mathbf{H}_{nqkp}}{(qn_e - pn_s)}$$

where n_s is the satellite mean motion.

Perturbations in orbital elements from tesseral harmonics are calculated by taking derivatives with respect to the elements. The expressions for the inclination and eccentricity functions were analyzed and a technique for their fast computation was chosen.

Eccentricity functions are expressed as follows. The limitation $e < 0.05$ allows limiting the calculation of these functions and their derivatives to the first order infinitesimal in e.

The inclination functions are calculated according to

$$A_{nq}^k = s^{|k-q|} c^{|k|} B_{nq}^k(\theta)$$

where

$$s = \sin(i/2), \qquad c = \cos(i/2), \qquad \theta = \cos i$$

For the chosen perturbation threshold, 49 A_{nq}^k functions turned out to be significant. Among them, 35 of the B_{nq}^k factors are in the form

$$B_{nq}^k = a + bx + cy$$

where a, b, and c are constants; x and y are terms of the θ, θ^2, and θ^3 type. The remaining 14 functions are more sophisticated, but calculation of this set of inclination functions by explicit formulas is about 1.5 times faster than using known recursive formulas [6].

To reduce the AP algorithm's CPU time, a special mode is introduced in which all perturbation amplitudes from tesseral harmonics are calculated once and remembered. For further accesses with the same element sets, only trigonometric factors are recalculated. In total, 284 amplitudes are calculated. The details of all equations used in the AP model are provided in Appendix B.

Numerical–Analytical (Combined) Prediction Algorithm

We consider the following system of motion equations:

$$\frac{d\mathbf{w}}{dt} = \mathcal{F}\frac{\partial \mathcal{H}(\mathbf{w})^T}{\partial \mathbf{w}} + \mathbf{F}(\mathbf{w}, h, t) \qquad (7)$$

where $\mathbf{F}$ is the generalized force of atmospheric drag given by [5]

$$\mathbf{F} = -S\rho(\mathbf{w}, h, t)\sqrt{\xi(\mathbf{w})}\mathbf{f}(\mathbf{w})$$

where ρ is the atmospheric density, h is the satellite altitude,

$$\xi = \frac{\mu}{p}\sqrt{1 + 2e\cos\vartheta + e^2}, \qquad \mathbf{f}_L = L\frac{(1 + 2e\cos\vartheta + e^2)}{(1 - e^2)}$$

$$\mathbf{f}_G = G, \qquad \mathbf{f}_H = H, \qquad \mathbf{f}_l = 2e\sin E + \frac{2}{e}\sqrt{1 - e^2}\sin\vartheta$$

$$\mathbf{f}_g = -\frac{2}{e}\sin\vartheta, \qquad \mathbf{f}_h = 0$$

and the atmospheric rotation is accounted for by matching ballistic coefficient corrections.

The atmospheric density is approximated by

$$\rho(\mathbf{w}, h, t) = \rho_\pi(\mathbf{w}, t)\exp\left((h_\pi - h)/H\right)(1 + F\cos(u - u^*))$$

where ρ_π is the density with regard to perigee, h_π is the perigee altitude, H is the scale height, and $F\cos(u - u^*)$ is the term characterizing solar bulging.

According to Brouwer and Clemence [5], we perform the transition from variables $\mathbf{w}$ to doubly averaged elements $\mathbf{w}''$:

$$\frac{d\mathbf{w}''}{dt} = \mathcal{F}\frac{\partial \mathcal{H}^{**}(\mathbf{w}'')^T}{\partial \mathbf{w}''} + \frac{\partial \mathbf{w}''}{\partial \mathbf{w}}\mathbf{F}(\mathbf{w}, h, t) \qquad (8)$$

In Eq. (8), we shall set

$$\frac{\partial \mathbf{w}''}{\partial \mathbf{w}} \approx \mathbf{E}$$

$\mathbf{F}(\mathbf{w}, h, t) \approx \mathbf{F}(\mathbf{w}'', h, t)$, and we will calculate h, taking into account perturbations from C_{20} and Earth oblateness, and we will employ the averaging technique to the derived system. Thus, the secular equation for triply averaged variables $\mathbf{w}'''$ is as follows:

$$\frac{d\mathbf{w}'''}{dt} = \mathcal{F}\frac{\partial \mathcal{H}^{**}(\mathbf{w}''')^T}{\partial \mathbf{w}'''} + \mathbf{Q}(\mathbf{w}''', t) \qquad (9)$$

where

$$\mathbf{Q}(\mathbf{w}''', t) = \frac{1}{2\pi}\int_0^{2\pi}\mathbf{F}(\mathbf{w}'', h, t)dl''$$

Formulas for the vector $\mathbf{Q}$ determination, after dropping terms of e^3 and $e^2\rho$ orders, are given in Appendix C.

The system of Eqs. (9) is integrated using the Runge–Kutta technique. The integration step, depending on the remaining lifetime, varies from 1 to 5 days. Thus, for the mode of joint processing of observations, the whole interval is covered in one integration step.

As the numerical integration with the Runge–Kutta technique is undertaken, we encounter two major difficulties. First, as the moment of the decay approaches, the acceleration of the atmospheric friction increases, and thus the step size, which permitted an acceptable accuracy in the beginning, is no longer good enough.

In the case of our algorithm, this issue is addressed by using a pair of Runge–Kutta formulas of the fourth and third orders (with a constant step). The algorithm of choosing the step and transferring from one algorithm to the other was found empirically and is not considered here (this is the field of responsibility of the complex algorithm).

Second, if the integration step is large, we will need a large number of the position vectors at different moments of time, whereas the Runge–Kutta algorithm produces only the initial and the final positions at a step. To deal with this problem, we have built interpolation formulas discussed subsequently.

One can obtain simple third-order interpolation formulas by using the value of the function that is calculated at the end of the step anyway. Then we have four known vectors at the step,

$$\mathbf{w}'''(t_0), \quad \mathbf{k_1}(t_0) = \frac{d\mathbf{w}'''}{dt}(\mathbf{w}_0''', t_0), \quad \mathbf{w}'''(t_1), \quad \mathbf{k_1}(t_1) = \frac{d\mathbf{w}'''}{dt}(\mathbf{w_0}''', t_1)$$

and we can use them to find the coefficients of the third-degree polynomial:

$$\mathbf{w}'''(\tau) = \mathbf{w_0}'''(t_0) + \mathbf{k}_1\tau + \mathbf{k}_2\tau^2 + \mathbf{k}_3\tau^3$$

where

$$\mathbf{k_2} = 3(\mathbf{w}'''(t_1) - \mathbf{w}'''(t_0)) - 2\mathbf{k}_1(t_0) - \mathbf{k}_1(t_1)$$

$$\mathbf{k_3} = 2(\mathbf{w}'''(t_1) - \mathbf{w}'''(t_0)) - 3\mathbf{k}_1(t_0) + \mathbf{k}_1(t_1)$$

For the fourth-order Runge–Kutta technique, one does not have to employ the value of the vector of the right-hand sides at the end of the step, and there is still the freedom of building the third-order formula (although the fourth-order formula cannot be obtained without additional data).

We used the fourth-degree formula of the form

$$\mathbf{w}'''(\tau) = \mathbf{w_0}''' + (\tfrac{1}{6}\mathbf{k_1} + \tfrac{1}{3}(\mathbf{k_2} + \mathbf{k_3}) + \tfrac{1}{6}\mathbf{k_4})\Delta t$$

$$\mathbf{k}_1 = \frac{d\mathbf{w}'''}{dt}(\mathbf{w_0}''', t_0), \qquad \mathbf{k}_2 = \frac{d\mathbf{w}'''}{dt}\left(\mathbf{w}_0''' + \frac{\mathbf{k}_1}{2}\Delta t, t_0 + \frac{\Delta t}{2}\right)$$

$$\mathbf{k}_3 = \frac{d\mathbf{w}'''}{dt}\left(\mathbf{w}_0''' + \frac{\mathbf{k}_2}{2}\Delta t, t_0 + \frac{\Delta t}{2}\right)$$

$$\mathbf{k}_4 = \frac{d\mathbf{w}'''}{dt}(\mathbf{w}_0''' + \mathbf{k}_3\Delta t, t_0 + \Delta t)$$

and we are looking for the polynomial of the form

$$\mathbf{w}'''(\tau) = \mathbf{w_0}''' + (q_1\mathbf{k}_1 + q_2(\mathbf{k_2} + \mathbf{k_3}) + q_3\mathbf{k_4})\Delta t$$

where the polynomials are written as

$$q_1 = \frac{2}{3}\nu^3 - \frac{3}{2}\nu^2 + \nu, \qquad q_2 = -\frac{2}{3}\nu^3 + \nu^2$$
$$q_3 = \frac{2}{3}\nu^3 - \frac{1}{2}\nu^2, \qquad \nu = \frac{(\tau - t_0)}{\Delta t}$$

Note that these formulas were obtained circa 1980. Several investigators have worked extensively in the direction of controlling the step size and interpolation inside the steps during the last 20 years. Apparently, it is no longer necessary to invent one's own formulas, and one should rather apply one of the Runge–Kutta techniques that are known to work well, with control of the step and interpolation inside the range (these techniques are also known as continuous numerical methods for ordinary differential equations with defect control).

Knowing $\mathbf{w}'''(t)$, the rectangular coordinates are found as done with procedure A. As a result, for joint processing of observations in the ongoing calculations mode, the CPU time of the NA procedure is close to that of the A procedure.

The accuracy of the NA algorithm was evaluated for different satellites by comparing with a numerical propagator. For the majority of the sample, the error in λ did not exceed 1–3% of the value of the atmospheric drag. The error in λ of the polynomial fit within one Runge–Kutta step does not exceed 0.3% of the value of atmospheric drag.

Fast approximation of the atmospheric density, documented by Vallado [7], is used in the algorithm. Using this approximation, only 150 operations are needed for one calculation of density. The formulas are given in Appendix I. The details of all equations used in the NA model are provided in Appendix C.

Numerical Prediction Algorithm

The task of obtaining the optimum calculation speed for orbits with tolerable eccentricities was posed for development of the N algorithm. The accuracy was set to $|\Delta r| \approx 8$ km/day. For this accuracy, the fastest method was the Adams method for nonsingular elements CEL.

In conventional notation, the method is as follows. The basic formula is

$$P_k E_1 C_{k+1} E_2 C_{k+1} \quad (k = 6, 10)$$

and the reserve formula is

$$P_k (E_1 C_{k+1})^t \quad (t < 5)$$

where P is the predicting formula and C is the correcting formula.

Main Characteristics of the Package

Table 1 presents general information regarding the methods and input CEL employed in the basic prediction procedures and an evaluation of the main components of the propagation errors. For each algorithm, Table 2 provides the following:

1) The average number of operations is in brackets for the complete procedure and without brackets for the ongoing calculations mode.

2) The CPU time for the Elbrus-2 computer (3.5 million operations/second processing speed).

3) The precision along the track is in brackets for the normal direction.

Table 3 presents the integral position determination precision (depending on both the accuracy of the element set, determined on the basis of observations, and propagation precision) for various types of orbits comprising the SSC catalog.

Issues of Element-Set Conversion

Conversion of CEL to REL is fulfilled in two stages. First, we determine t_Ω using the condition $u(t_\Omega) = 0$. Then osculating

Table 1 Methods and catalog element sets employed in the prediction procedures

Procedure	Catalog element set	Method	Precision
A	$W'' = \{\lambda'', L'', \theta'', \Omega'', h'', k'', S\}$	$W(t) = W''(t) + \delta\tilde{W}(t) + \delta W(t) + \delta W_{\text{tes}}(t)$	$\Delta W'' \approx \mathcal{O}(c_{20}^2 t)$, $\delta\tilde{W}\,\delta W \approx \mathcal{O}(c_{20})$, $\delta W \approx \mathcal{O}(\frac{c_{n0}}{c_{20}})$, $\delta W_{\text{tes}} \approx \mathcal{O}(\frac{c_{22}}{n_E} e^2)$
AP	$W'' = \{\lambda'', L'', \theta'', \Omega'', h'', k'', S\}$	$W(t) = W''(t) + \delta\tilde{W}(t) + \delta W(t) + \delta W_{\text{tes}}(t)$	$\delta W_{\text{tes}} \approx \mathcal{O}(\frac{c_{nm}}{n_E} e^2)$
NA	$W''' = \{\lambda''', L''', \theta''', \Omega''', h''', k''', S\}$	$\frac{d\mathbf{w}'''}{dt} = \mathcal{F}\frac{\partial\mathcal{H}^{**}(\mathbf{w}''')^T}{\partial\mathbf{w}'''} + \mathbf{Q}(\mathbf{w}''', t)$	$Q \approx \mathcal{O}(e^3, e^2\rho)$, $\Delta\lambda_{\text{rel}} \approx 3\%$
N	$W = \{\lambda, L, \theta, \Omega, h, k, S\}$	Base $P_k E_1 C_{k+1} E_2 C_{k+1}$ $(k = 6, 10)$, reserve $P_k(E_1 C_{k+1})^t$, $t \le 4$	$\Delta r \approx 8$ km/day
NP	$W = \{\lambda, L, \theta, \Omega, h, k, S\}$	U	$\Delta r \approx 0.1$ km/day

Table 2 Perturbation factors, number of operations, accuracy of the different prediction procedures

Procedures	Earth	Atmosphere	Number of operations	CPU time	Precision
A	8×2	Static model $F_{10.7}$	1100 (2200)	0.0003 s (0.0006 s)	1.5 (0.5) km, $e < 0.05$
AP	8×8	Static model $F_{10.7}$	2800 (6300)	0.0006 s (0.0015 s)	0.2 (0.1) km, $e < 0.05$
NA	8×8	Dynamic model	4200 (12,000)	0.001 s (0.003 s)	0.2 (0.1) km, $e < 0.05$, 3% atmospheric drag, $e < 0.1$
N	6×2	Dynamic model	10^6	0.16 s/day	8 km/day
NP	14×14	Dynamic model	$(0.6–2.7) \times 10^6$ (Earth 8×8)	0.1–0.45 s/day	0.1 km

Table 3 Integral prediction accuracy (in kilometers)

	Interval of prediction (days)						
Class	1	2	3	5	7	10	Percent of objects
$e > 0.05$	2.0	4.2	7.7	15	29	60	12
$e < 0.05$, $H < 400$ km	8.6	27	65	——	——	——	1
$e < 0.05$, $400 < H < 1500$ km	1.0	1.7	2.7	4.7	7.0	15	82
$e < 0.05$, $H > 1500$ km	0.9	1.3	1.6	1.9	2.7	3.4	5

elements e, i, Ω, and ω are determined with respect to this time. The draconic period with the first order of accuracy in C_{20} is found using the well-known equation [2] or by Eq. (1), and ΔT is determined by Eq. (2). Thus, the first stage is finished. If the element set was determined with high accuracy, based on observations for a long period of time, we will have accuracy losses in calculating T_Ω. That is why the secular component of T_Ω is calculated at the second stage with the second order of accuracy in C_{20}.

The issue of conversion of catalog elements to NORAD elements [8] and vice versa is of certain interest. Regarding the basic technique, our algorithms A, AP, and NA and NORAD procedures use similar doubly averaged elements. But for calculation of the gravitational perturbations, different models are used, the same factors are treated differently, and atmosphere models are uncorrelated. Thus, if the accuracy losses are undesirable, propagation of the elements must be performed with the same prediction procedures that were employed in their determination.

If it is not possible to use the appropriate algorithm, the issue of element-set coordination arises. First, errors of secular character must be minimized. Prediction errors of a periodic character can be neglected because they are relatively small and limited. This approach will allow each center to not require the complete prediction algorithm of its partner, but to require only a rather small part describing secular motion.

Catalog Element Sets and the Technique of Their Prediction

The package of prediction procedures used in satellite catalog maintenance is described. The most frequently used analytical and numerical–analytical (combined) procedures are considered in detail. Catalog element sets consist of the averaged orbit elements $\{t, \lambda, L, \theta, \Omega, h, k, \text{and } S\}$. In Russia, the well-known and frequently used RELs [2] consisting of the time of equator crossing, draconic period, period variation per one revolution, and Kepler's osculating elements at the ascending node can be derived from the catalog element sets by means of special conversion procedures similar to those described by Nazarenko [9]. Naturally, the best prediction accuracy can be attained using the catalog element sets and the same prediction algorithm that was employed for element-set determination. Otherwise, additional conversions and transformations are needed that obviously do not increase prediction accuracy.

The prediction-algorithm package contains the following basic procedures: 1) analytical prediction algorithm (A), 2) analytical prediction algorithm with enhanced accuracy (AP), 3) numerical–analytical (combined) prediction algorithm with enhanced accuracy (NA), 4) numerical prediction algorithm (N), and 5) numerical prediction algorithm with enhanced accuracy (NP).

Procedure A is used for the tasks of correlation and detection of observations. For element-set updating, procedures AP, NA, N, and NP are used. For 90% of the objects, procedure AP is employed, and for the others, the NA procedure is used, except for objects close to reentering and individual satellites with special requirements for orbit determination and prediction. For the former, the procedure N is employed, and for the latter, the NP procedure is employed.

Each of the prediction procedures has its own identifier that is stored in the catalog, together with the updated element sets for each satellite. The criteria for choosing the identifier of the prediction procedure include accounting for the value of atmospheric drag, perigee altitude, orbital eccentricity, and remaining lifetime. For the correct choice of the procedure to propagate the catalog element sets, we recommend retrieving the identifier of the prediction procedure from the catalog, together with the element sets.

It is worth noting that, depending on the identifier of the prediction procedure, variously averaged orbital elements are stored in the catalog. For the AP and NA algorithms, short-periodic perturbations from the second zonal harmonic, long-periodic perturbations from the zonal harmonics up to the eighth order, and the most significant periodic perturbations from tesseral harmonics up to the eighth order are excluded from the element sets. The solution is obtained using Brouwer's technique [5] and the resulting elements are doubly averaged element sets, similar to NORAD [9] TLEs. The NA procedure, in accounting for atmospheric perturbations, employs an additional averaging over the fast variable of doubly averaged Brouwer elements that results in an additional difference between the element sets produced by NA and AP, respectively.

When using the numerical algorithms N or NP, osculating elements are updated and stored in the catalog. The N algorithm takes into account gravitational perturbations from the 6×2 field and atmospheric drag according to the dynamic model [7]. The NP algorithm takes into account gravitational perturbations from the 8×8 field, atmospheric drag, and gravitational lunar and solar perturbations.

The matching ballistic coefficient[†] S ($\text{km}^3/(\text{kg} \cdot \text{s}^2)$) used in the analytical and combined models also has a rather different sense, because the analytical model [7] uses a simplified static approximation of the density of the night atmosphere, depending only on the mean (for preceding 135 days) level of solar activity. As a result, even for slight atmospheric drag, its value, calculated by different algorithms using the same initial data, may differ by a factor of 1.5 to 2.

On the contrary, the matching ballistic coefficient for the numerical and combined procedures will be close. The averaging of the model's density approximation employed in the NA procedure leads to methodical errors not exceeding 3% of the value of the atmospheric drag in the realm of this procedure application.

In subsequent sections, the formulas of the A, AP, and NA models as well as the basic auxiliary procedures are described in detail. Every effort has been made to maintain a parallel structure between the equations and the computer code. In particular, for many cases, the equality sign = is used not in a conventional arithmetical sense, but to denote assigning to the left-side variable the value of the expression on the right side. The final section presents the list of the most frequently used constants and their values.

Appendix A: Analytic Model

The mean catalog element sets can be propagated analytically using the A algorithm.

Input parameters are as follows: t is the epoch of the element set; λ'', L'', θ'', Ω'', h'', and k'' are the catalog element set at epoch (doubly averaged elements $\mathbf{w}''$); S is the matching ballistic coefficient; t_n is the time of prediction; and A is the array of constants defining the status of the atmosphere.

Output parameters are as follows: X, Y, Z, $\dot{X}$, $\dot{Y}$, and $\dot{Z}$ are the components of the position and velocity vectors [in geocentric stationary coordinate system (GSCS) for time t_n]; λ_n, L_n, θ_n, Ω_n, h_n, and k_n are the osculating element set for time t_n.

The secular effects of the Earth zonal harmonics are computed using the equations described in Appendix D. The critical inclination flag P_{cr} is based on the inclination of the satellite. In addition to several initialization constants, the results of those calculations provide $\dot{l}$, $\dot{g}$, $\dot{\Omega}$, and P_{cr}.

The secular effects of the atmospheric drag are computed using the equations described in Appendix E. The results of those calculations provide the following constants and rate terms: $\ddot{l}$, $\ddot{g}$, $\ddot{\Omega}$, $\dot{L}$, $\dot{e}$, x, and z_3. The coefficients of the long-period periodic terms are computed using the equations described in Appendix F. The results of those calculations provide j_Ω, j_{eg}, and j_λ and A_{1j}, A_{2j}, A_{3j}, and A_{4j}, where $j = 1, \ldots, 4$.

If $P_{\text{cr}} = 1$, then coefficients A_i $(i = 1, \ldots, 4)$ are computed in the following sequence:

[†]Historically, the Russian catalog uses weight in kilograms. Thus, the mass is the weight divided by the acceleration of gravity g. This is the reason for the nonstandard unit dimensions of the ballistic coefficient.

$$\sin j\varphi_0 = \sin\varphi_0\cos(j-1)\varphi_0 + \sin(j-1)\varphi_0\cos\varphi_0$$

$$\cos j\varphi_0 = \cos\varphi_0\cos(j-1)\varphi_0 - \sin(j-1)\varphi_0\sin\varphi_0 \quad (j = 2,\ldots,4)$$

$$A_1 = \sum_{j=1}^{4} A_{1j} j \sin j\varphi_{0,} \qquad A_2 = \sum_{j=1}^{4} A_{2j} j \cos j\varphi_0$$

$$A_3 = \sum_{j=1}^{4} A_{3j} j \cos j\varphi_{0,} \qquad A_4 = \sum_{j=1}^{4} A_{4j} j \cos j\varphi_0$$

Secular evolution is calculated using the formulas

$$\Delta t = 86{,}400(t_n - t), \quad l'' = l_0 + \dot{l}\Delta t, \quad \Omega'' = \Omega_0 + \dot{\Omega}\Delta t$$

$$g'' = g_0 + \dot{g}\Delta t, \quad L'' = L, \quad e'' = e, \quad \eta'' = \eta, \quad \theta'' = \theta, \quad s'' = s$$

Atmospheric drag terms are calculated as follows:

$$y = x\Delta t$$

If $|y| \geq 0.1$, then

$$z_4 = \begin{cases} \ell n(1-y) & y < 0.95 \\ -3 & y \geq 0.95 \end{cases}, \quad z_1 = 2(y + (1-y)z_4)/y^2, \quad z_2 = -z_4/y$$

If $|y| < 0.1$, then

$$z_1 = z_2 = 1, \qquad \Delta e_a = \dot{e} z_2 z_3 \Delta t, \qquad l'' = l'' + 0.5\ddot{l} z_1 \Delta t^2$$

$$g'' = g'' + \ddot{g} z_1 \Delta t^2, \qquad \Omega'' = \Omega'' + \ddot{\Omega} z_1 \Delta t^2$$

$$L'' = L'' + \dot{L} z_2 \Delta t, \qquad e'' = e'' + \Delta e_a, \qquad \eta'' = \eta - e\eta_1 \Delta e_a$$

$$\lambda'' = l'' + g'', \qquad h'' = e''\cos\varphi, \qquad k'' = e''\sin\varphi$$

where $\sin\varphi = \cos g''$ and $\cos\varphi = \sin g''$.

For the critical inclination ($P_{\text{cr}} = 1$), long-period perturbations (averaged elements $\mathbf{w}'$) are transformed into secular and are added according to the formulas

$$\Delta e' = A_1 \Delta t, \qquad \Delta\Omega' = A_2 \Delta t + j_\Omega \Delta e' \Delta t$$

$$e''\Delta g' = A_3 \Delta t + j_{eg} \Delta e' \Delta t, \qquad \Delta\lambda' = A_4 \Delta t + j_\lambda \Delta e' \Delta t$$

$$\Delta i' = -\xi_8 \Delta e' \qquad e'' = e'' + \Delta e', \qquad \eta'' = \eta'' - e\eta_1 \Delta e'$$

$$\theta'' = \theta'' - s\Delta i', \qquad \Omega'' = \Omega'' + \Delta\Omega'$$

$$h'' = h'' + \Delta e' \cos\varphi + e''\Delta g' \sin\varphi$$

$$k'' = k'' + \Delta e' \sin\varphi - e''\Delta g' \cos\varphi, \qquad \lambda'' = \lambda'' + \Delta\lambda'$$

$$s'' = s'' + \theta\Delta i', \qquad e'' = 0.5(e'' + (h''^2 + k''^2)/e'')$$

$$\sin\varphi = k''/e'', \qquad \cos\varphi = h''/e'', \qquad g'' = \tan^{-1}(\cos\varphi, \sin\varphi)$$

$$l'' = \lambda'' - g''$$

For the case $P_{\text{cr}} = 0$, the long-period periodics are computed using the formulas

$$\sin j\varphi = \sin\varphi\cos(j-1)\varphi + \sin(j-1)\varphi\cos\varphi$$

$$\cos j\varphi = \cos\varphi\cos(j-1)\varphi - \sin(j-1)\varphi\sin\varphi \quad (j = 2,\ldots,4)$$

$$\Delta e' = \sum_{j=1}^{4} A_{1j}\cos j\varphi, \qquad \Delta\Omega' = \sum_{j=1}^{4} A_{2j}\sin j\varphi$$

$$e''\Delta g' = \sum_{j=1}^{4} A_{3j}\sin j\varphi, \qquad \Delta\lambda' = \sum_{j=1}^{4} A_{4j}\sin j\varphi \qquad \Delta i' = -\xi_8 \Delta e'$$

Solve Kepler's equation and compute the short-period periodics using the equations described in Appendix G. The results of those calculations provide ΔL, Δe, $e\Delta g$, $\Delta\lambda$, $\Delta\Omega$, and Δi.

The periodics from tesseral harmonics are computed using the formulas

$$\delta i = \gamma_{22} s \cos 2\tilde{\Omega}, \qquad \delta\Omega = -\gamma_{22}\theta \sin 2\tilde{\Omega}$$

$$\delta g = -\theta\delta\Omega - 1.5\gamma_{22} b_{20} \sin 2\tilde{\Omega}, \qquad \delta l = -1.5\gamma_{22}\eta b_{20} \sin 2\tilde{\Omega}$$

where

$$\cos 2\tilde{\Omega} = C_{22}\cos 2(\Omega - S_{\text{star}}) + S_{22}\sin 2(\Omega - S_{\text{star}})$$

$$\sin 2\tilde{\Omega} = C_{22}\sin 2(\Omega - S_{\text{star}}) - S_{22}\cos 2(\Omega - S_{\text{star}})$$

$$\gamma_{22} = 1.5\gamma\left(\frac{\sqrt{\mu R_e}}{L\eta}\right)^4, \qquad \gamma = \frac{\mu^2}{n_e L^3}$$

The C and S tesseral and sectoral coefficients are given in Table H1. Total increments of the elements are calculated as follows:

$$\Delta i = \Delta i + \delta i, \qquad \Delta\Omega = \Delta\Omega + \delta\Omega$$

$$e\Delta g = e\Delta g + e\delta g, \qquad \Delta\lambda = \Delta\lambda + \delta l + \delta g$$

If the inclination is not critical ($P_{\text{cr}} = 0$), then the previously computed long-period periodics are added to them:

$$\Delta e = \Delta e + \Delta e', \qquad \Delta i = \Delta i + \Delta i', \qquad \Delta\Omega = \Delta\Omega + \Delta\Omega'$$

$$e\Delta g = e\Delta g + e''\Delta g', \qquad \Delta\lambda = \Delta\lambda + \Delta\lambda'$$

Perturbations of h and k elements are calculated as

$$\Delta h = \Delta e\cos\varphi + e\Delta g\sin\varphi, \qquad \Delta k = \Delta e\sin\varphi - e\Delta g\cos\varphi$$

Osculating elements are calculated according to

$$L_n = L'' + \Delta L, \qquad \theta_n = \theta'' - s''\Delta i, \qquad \lambda_n = \lambda'' + \Delta\lambda$$

$$\Omega_n = \Omega'' + \Delta\Omega, \qquad h_n = h'' + \Delta h, \qquad k_n = k'' + \Delta k$$

The angles λ_n and Ω_n are brought to the $[0, 2\pi)$ interval. Finally, position and velocity are calculated using the equations in Appendix L.

Appendix B: Analytic Propagator Model

The mean catalog element sets can be propagated analytically using the AP algorithm.

Input parameters are as follows: t is the epoch of the element set; λ'', L'', θ'', Ω'', h'', and k'' are the catalog element set at epoch (doubly averaged elements $\mathbf{w}''$); S is the matching ballistic coefficient; t_n is the time of prediction; and A is the array of constants defining the status of the atmosphere.

Output parameters are as follows: X, Y, Z, $\dot{X}$, $\dot{Y}$, and $\dot{Z}$ are the components of the position and velocity vectors (in GSCS for time t_n); λ_n, L_n, θ_n, Ω_n, h_n, and k_n are the osculating element set for time t_n.

The secular effects of the Earth zonal harmonics are computed using the equations described in Appendix D. The critical inclination flag P_{cr} is based on the inclination of the satellite. In addition to several initialization constants, the results of those calculations provide $\dot{l}$, $\dot{g}$, $\dot{\Omega}$, and P_{cr}. The secular effects of the atmospheric drag are computed using the equations described in Appendix E. The results of those calculations provide the following constants and rate terms: $\ddot{l}$, $\ddot{g}$, $\ddot{\Omega}$, $\dot{L}$, $\dot{e}$, x, and z_3. The coefficients of the perturbations from the tesseral harmonics are calculated by access to the BAX1 subroutine, described in Appendix J.

The coefficients of the long-period periodic terms are computed using the equations described in Appendix F. The results of those calculations provide j_Ω, j_{eg}, and j_λ and A_{1j}, A_{2j}, A_{3j}, and A_{4j}, where $j = 1,\ldots,4$.

If $P_{\text{cr}} = 1$, then coefficients A_i ($i = 1,\ldots,4$) are computed in the following sequence:

$$\sin j\varphi_0 = \sin\varphi_0 \cos(j-1)\varphi_0 + \sin(j-1)\varphi_0 \cos\varphi_0$$

$$\cos j\varphi_0 = \cos\varphi_0 \cos(j-1)\varphi_0 - \sin(j-1)\varphi_0 \sin\varphi_0 \quad (j = 2, \ldots, 4)$$

$$A_1 = \sum_{j=1}^{4} A_{1j} j \sin j\varphi_{0,} \qquad A_2 = \sum_{j=1}^{4} A_{2j} j \cos j\varphi_{0,}$$

$$A_3 = \sum_{j=1}^{4} A_{3j} j \cos j\varphi_{0,} \qquad A_4 = \sum_{j=1}^{4} A_{4j} j \cos j\varphi_0$$

Secular evolution is calculated using the formulas

$$\Delta t = 86{,}400(t_n - t), \quad l'' = l_0 + \dot{l}\Delta t, \quad \Omega'' = \Omega_0 + \dot{\Omega}\Delta t$$

$$g'' = g_0 + \dot{g}\Delta t, \quad L'' = L, \quad e'' = e, \quad \eta'' = \eta, \quad \theta'' = \theta, \quad s'' = s$$

Atmospheric drag terms are calculated as follows:

$$y = x\Delta t$$

If $|y| \geq 0.1$, then

$$z_4 = \begin{cases} \ell n(1-y) & y < 0.95 \\ -3 & y \geq 0.95 \end{cases}, \quad z_1 = 2(y + (1-y)z_4)/y^2, \quad z_2 = -z_4/y$$

If $|y| < 0.1$, then

$$z_1 = z_2 = 1, \qquad \Delta e_a = \dot{e} z_2 z_3 \Delta t, \qquad l'' = l'' + 0.5\ddot{l} z_1 \Delta t^2$$

$$g'' = g'' + \ddot{g} z_1 \Delta t^2, \qquad \Omega'' = \Omega'' + \ddot{\Omega} z_1 \Delta t^2$$

$$L'' = L'' + \dot{L} z_2 \Delta t, \qquad e'' = e'' + \Delta e_a, \qquad \eta'' = \eta - e\eta_1 \Delta e_a$$

$$\lambda'' = l'' + g'', \qquad h'' = e'' \cos\varphi, \qquad k'' = e'' \sin\varphi$$

where $\sin\varphi = \cos g''$ and $\cos\varphi = \sin g''$.

For the critical inclination ($P_{\rm cr} = 1$), the long-period perturbations are transformed into secular and are added according to the formulas

$$\Delta e' = A_1 \Delta t, \qquad \Delta\Omega' = A_2 \Delta t + j_\Omega \Delta e' \Delta t$$

$$e'' \Delta g' = A_3 \Delta t + j_{eg} \Delta e' \Delta t, \qquad \Delta\lambda' = A_4 \Delta t + j_\lambda \Delta e' \Delta t$$

$$\Delta i' = -\xi_8 \Delta e' \qquad e'' = e'' + \Delta e', \qquad \eta'' = \eta'' - e\eta_1 \Delta e'$$

$$\theta'' = \theta'' - s\Delta i', \qquad \Omega'' = \Omega'' + \Delta\Omega'$$

$$h'' = h'' + \Delta e' \cos\varphi + e'' \Delta g' \sin\varphi$$

$$k'' = k'' + \Delta e' \sin\varphi - e'' \Delta g' \cos\varphi, \qquad \lambda'' = \lambda'' + \Delta\lambda'$$

$$s'' = s'' + \theta \Delta i', \qquad e'' = 0.5(e'' + (h''^2 + k''^2)/e'')$$

$$\sin\varphi = k''/e'', \qquad \cos\varphi = h''/e''$$

$$g'' = \tan^{-1}(\cos\varphi, \sin\varphi), \qquad l'' = \lambda'' - g''$$

For the case $P_{\rm cr} = 0$, the long-period periodics are computed using the formulas

$$\sin j\varphi = \sin\varphi \cos(j-1)\varphi + \sin(j-1)\varphi \cos\varphi$$

$$\cos j\varphi = \cos\varphi \cos(j-1)\varphi - \sin(j-1)\varphi \sin\varphi \quad (j = 2, \ldots, 4)$$

$$\Delta e' = \sum_{j=1}^{4} A_{1j} \cos j\varphi, \qquad \Delta\Omega' = \sum_{j=1}^{4} A_{2j} \sin j\varphi$$

$$e'' \Delta g' = \sum_{j=1}^{4} A_{3j} \sin j\varphi, \qquad \Delta\lambda' = \sum_{j=1}^{4} A_{4j} \sin j\varphi, \qquad \Delta i' = -\xi_8 \Delta e'$$

Solve Kepler's equation and compute the short-period periodics using the equations described in Appendix G. The results of those calculations provide ΔL, Δe, $e\Delta g$, $\Delta\lambda$, $\Delta\Omega$, and Δi.

The periodics from tesseral harmonics are calculated as described in Appendix K. The results of those calculations provide δL, δe, $e\delta g$, $\delta\lambda$, $\delta\Omega$, and δi. Total increments of the elements are calculated as follows:

$$\Delta L = \Delta L - \delta L, \qquad \Delta e = \Delta e - \delta e, \qquad \Delta i = \Delta i - \delta i$$

$$\Delta\Omega = \Delta\Omega - \delta\Omega, \qquad e\Delta g = e\Delta g - e\delta g, \qquad \Delta\lambda = \Delta\lambda - \delta\lambda$$

If the inclination is not critical ($P_{\rm cr} = 0$), then the previously computed long-period periodics are added to them:

$$\Delta e = \Delta e + \Delta e', \qquad \Delta i = \Delta i + \Delta i', \qquad \Delta\Omega = \Delta\Omega + \Delta\Omega'$$

$$e\Delta g = e\Delta g + e''\Delta g', \qquad \Delta\lambda = \Delta\lambda + \Delta\lambda'$$

Perturbations of h and k elements are calculated as

$$\Delta h = \Delta e \cos\varphi + e\Delta g \sin\varphi, \qquad \Delta k = \Delta e \sin\varphi - e\Delta g \cos\varphi$$

Osculating elements are calculated according to

$$L_n = L'' + \Delta L, \qquad \theta_n = \theta'' - s''\Delta i, \qquad \lambda_n = \lambda'' + \Delta\lambda$$

$$\Omega_n = \Omega'' + \Delta\Omega, \qquad h_n = h'' + \Delta h, \qquad k_n = k'' + \Delta k$$

The angles λ_n and Ω_n are brought to the $[0, 2\pi)$ interval. Finally, position and velocity are calculated using the equations in Appendix L.

Appendix C: Numerical–Analytical Algorithm

The mean catalog element sets can be propagated by means of the numerical–analytical technique using algorithm NA.

Input parameters are as follows: $t_{\rm in}$ is the epoch of the element set; λ''', L''', θ''', Ω''', h''', and k''' are the catalog element set at epoch (triple averaged elements $\mathbf{w}'''$); S is the matching ballistic coefficient; $t_{\rm out}$ is the time of prediction; P_f is the flag indicating mode of operation; and MKA is the array of constants defining the status of the atmosphere.

Output parameters are as follows: for $P_f = 1$, X, Y, Z, $\dot{X}$, $\dot{Y}$, and $\dot{Z}$ are the components of the position and velocity vectors (in GSCS for time t_n); λ_n, L_n, θ_n, Ω_n, h_n, and k_n are the osculating element set for time t_n. For $P_f = 2$, $t_{\rm ch}$ is the time of decay.

The algorithm NA includes four internal modules: SB01 is the module of the differential equations of satellite motion integration using Runge–Kutta's procedure of the third order. SB02 is the module for calculating the right sides of the averaged differential equations of motion or the osculating element sets, position, and velocity (depending on the flag $P_{\rm tp}$). SB03 is the module for choosing the integration step. SB04 is the module of Bessel function calculation.

A description of the modules is given subsequently. The description of the algorithm itself follows the description of the modules.

SB01: Differential Equations Integration Module

Integration of the averaged differential equations is accomplished using the third-order Runge–Kutta procedure. The formula is as follows:

$$\mathbf{w}(t + h) = \mathbf{w}(t) + \tfrac{h}{8}(2\mathbf{K_1} + 3\mathbf{K_2} + 3\mathbf{K_3})$$

where $\mathbf{w}(t)$ is the element set (λ, L, θ, Ω, h, and k) vector for time t, h is the integration step, $\mathbf{w}(t+h)$ is the element-set vector after the integration step, $\mathbf{K_1}$ is the vector of right sides of the differential equations ($\dot{\lambda}$, $\dot{L}$, $\dot{\theta}$, $\dot{\Omega}$, $\dot{h}$, and $\dot{k}$) obtained by access to the module SB02 with elements $\mathbf{w}(t)$ for time t, $\mathbf{K_2}$ is the vector of right sides of the differential equations obtained by access to module SB02 with elements $\mathbf{w}(t) + \frac{2}{3}h\mathbf{K_1}$ for time $t + \frac{2}{3}h$, and $\mathbf{K}_3$ is the vector of right sides of the differential equations obtained by access to module SB02 with elements $\mathbf{w}(t) + \frac{2}{3}h\mathbf{K}_2$ for time $t + \frac{2}{3}h$.

In cases of failures within the module SB02 or the element set falling out of tolerable limits (for low reentering satellites), the failure flag $P_{\rm fin}$ is set to 1; otherwise, it is set to 0.

SB02: Module for Calculating Right Sides or Osculating Element Set and Position

Module SB02 is used to calculate the right sides of the averaged differential equations or the osculating element set, position, and velocity. The module has three flags of external control: P_{on} is the flag of first access to the module: certain preliminary quantities are calculated and registered at the first access that are not recalculated at further references to the module, and the value of the flag is 1 or 0; P_{cr} is the flag of the critical inclination: formulas for calculating right sides of the differential equations and long-period periodic terms of the zonal harmonics depend on this flag; the values are 1 for the critical inclination and 0 otherwise. P_{tp} is the flag of the operation mode: when $P_{\text{tp}} = 2$, the right sides of the averaged differential equations are determined from mean elements, and when $P_{\text{tp}} = 1$, the periodics, osculating elements, position, and velocity are calculated.

The module also elaborates the value of the failure flag: $P_{\text{fin}} = 1$ when the element set falls out of tolerable limits and $P_{\text{fin}} = 0$ otherwise.

Input parameters of the module are mean elements (λ, L, θ, Ω, h, and k) at time t. The formulas of the SB02 module are given subsequently.

The secular effects of the Earth zonal harmonics are computed using the equations described in Appendix D. Also, the critical inclination flag P_{cr} is based on the inclination of the satellite. In addition to several initialization constants, the results of those calculations provide $\dot{l}$, $\dot{g}$, $\dot{\Omega}$, and P_{cr}. In addition, the following rate terms are also needed:

$$\dot{\lambda} = \dot{l} + \dot{g}, \qquad \dot{h} = k\dot{g}, \qquad \dot{k} = -h\dot{g}, \qquad \dot{\theta} = 0, \qquad \dot{L} = 0$$

Further inclination and eccentricity functions are calculated to obtain coefficients of long-periodic terms. Calculations are performed for noncritical inclination ($P_{\text{cr}} = 0$) independently of the flag P_{tp} (because long-period periodics participate both in computing of the right sides of the atmospheric drag and in calculating osculating elements and position) and for the critical inclination ($P_{\text{cr}} = 1$) when $P_{\text{tp}} = 2$. (For critical inclination, the long-period terms turn to secular; they do not participate in the calculation of the atmospheric drag terms of the right sides and in computing the position, but they produce their own input to the right sides of the equations of motion.)

The coefficients of the perturbations from the tesseral harmonics are calculated by access to the BAX1 subroutine, described in Appendix J.

The coefficients of the long-period periodic terms are computed using the equations described in Appendix F. The results of those calculations provide j_Ω, j_{eg}, and j_λ and A_{1j}, A_{2j}, A_{3j}, and A_{4j}, where $j = 1, \ldots, 4$.

For the case of the critical inclination ($P_{\text{cr}} = 1$), long-period periodics are set to zero, and the additives to the right sides of the differential equations, calculated when $P_{\text{tp}} = 2$, are as follows:

$$\dot{e}_{\text{cr}} = -\sum_{j=1}^{4} A_{1j} j \sin j\varphi, \qquad \dot{\Omega}_{\text{cr}} = \sum_{j=1}^{4} A_{2j} j \cos j\varphi$$

$$e\dot{g}_{\text{cr}} = \sum_{j=1}^{4} A_{3j} j \cos j\varphi, \qquad \dot{\lambda}_{\text{cr}} = \sum_{j=1}^{4} A_{4j} j \cos j\varphi, \qquad \dot{\theta}_{\text{cr}} = \dot{e}_{\text{cr}} e\theta/\eta_2$$

$$\dot{h}_{\text{cr}} = \dot{e}_{\text{cr}} \cos\varphi + e\dot{g}_{\text{cr}} \sin\varphi, \qquad \dot{k}_{\text{cr}} = \dot{e}_{\text{cr}} \sin\varphi - e\dot{g}_{\text{cr}} \cos\varphi$$

For noncritical inclination ($P_{\text{cr}} = 0$), long-period periodics are as follows:

$$\Delta e' = \sum_{j=1}^{4} A_{1j} \cos j\varphi, \qquad \Delta\Omega' = \sum_{j=1}^{4} A_{2j} \sin j\varphi$$

$$e\Delta g' = \sum_{j=1}^{4} A_{3j} \sin j\varphi, \qquad \Delta\lambda' = \sum_{j=1}^{4} A_{4j} \sin j\varphi, \qquad \Delta i' = -\frac{e\theta}{\eta_2 s}\Delta e'$$

where $\sin\varphi = \cos g''$, $\cos\varphi = \sin g''$, $\sin j\varphi$, and $\cos j\varphi$ are calculated by multiple-angle formulas. Further, under $P_{\text{tp}} = 2$, atmospheric drag perturbations of the right sides of the differential equations are computed. The formula is

$$h_n = a(1 - e - \Delta e') - R_e + C_{20}\frac{R_e^2}{4a\eta_2}\left((1-3\theta^2)\left(1 + e\eta_{22} + \frac{2\eta}{1+e}\right) - B_{20}(2\cos^2\varphi - 1)\right) + R_e\alpha_{\text{er}}B_{20}\cos^2\varphi$$

where α_{er} is the Earth-oblateness coefficient.

If $h_n > 1500$, then we take $h_n = 1500$. If $h_n < 120$, then $H = 6$ is the scale height. If $120 \le h_n \le 1500$, then the index of the layer (defining the subarray of atmospheric parameters) is calculated, where constants from the atmosphere parameters array correspond to the fixed level of solar activity F_0.

By access to the atmosphere subroutine described in Appendix I, the following parameters are determined: $\bar{\rho}_0$ is the mean density; H is the scale height; and F, $\sin u^*$, $\cos u^*$ are the solar bulging parameters.

The input parameters are h_n, s, θ, Ω, t, $P_f = 1$, and the array MKA.

Further, the following variables are calculated:

$$A = \frac{C_{20}R_e^2}{4a\eta_2}, \quad C = -\frac{A(1-3\theta^2)}{H}, \quad R_1 = \frac{(A - 0.5R_e\alpha_{\text{er}})B_{20}}{H}$$

$$R_2 = -\eta_2(2\cos^2\varphi - 1), \quad R_3 = \frac{a\Delta e'}{H}, \quad R_4 = \frac{a(e\Delta\lambda' - e\Delta g')}{H}$$

$$R_1' = 0.5R_1\eta\sin 2\varphi, \quad R_2' = 0.5R_1R_2$$

$$\tilde{F}_1 = F(\cos u^* \sin\varphi + \sin u^* \cos\varphi)$$

$$\tilde{F}_2 = F\eta(\sin u^* \sin\varphi - \cos u^* \cos\varphi), \quad \bar{a}_0 = 1 - en_2 - 0.5R_4\tilde{F}_2$$

$$\bar{a}_1 = e(1 - R_2') + \tilde{F}_2R_1' + \tilde{F}_1(1 - R_2')$$

$$\bar{a}_2 = en_1(en_1 + R_3) - 2R_2' + 0.5R_4\tilde{F}_2$$

$$\bar{a}_3 = -3eR_2' - \tilde{F}_1R_2' - \tilde{F}_2R_1'$$

$$\bar{b}_1 = -(R_4 + R_1'e) + \tilde{F}_2(1 + R_2') + \tilde{F}_1R_1'$$

$$\bar{b}_2 = 2R_1' - eR_4 - 0.5R_4\tilde{F}_1, \quad \bar{b}_3 = 3R_1'e + \tilde{F}_1R_1' - \tilde{F}_2R_2'$$

$$z = \frac{ae}{H} + 2e(C + R_2') + R_3, \quad A = \frac{ae}{H} + 2(Ce - R_2') + R_3(1 + en_1)$$

By access to the SB04 procedure, Bessel's functions I_0, I_1, I_2, I_3, and I_4 are computed:

$$\Sigma_1 = \bar{a}_0I_0 + \bar{a}_1I_1 + \bar{a}_2I_2 + \bar{a}_3I_3$$

$$\Sigma_2 = \bar{a}_0I_1 + \tfrac{1}{2}(\bar{a}_1(I_0 + I_2) + \bar{a}_2(I_1 + I_3) + \bar{a}_3(I_2 + I_4))$$

$$\Sigma_3 = \tfrac{1}{2}(\bar{b}_1(I_0 - I_2) + \bar{b}_2(I_1 - I_3) + \bar{b}_3(I_2 - I_4))$$

$$\varepsilon_{\text{atm}} = -\frac{\bar{\rho}_0 e^{-A}\mu S}{G}, \qquad Q_e = 2\varepsilon_{\text{atm}}\eta_2\Sigma_2, \qquad Q_g = 2\varepsilon_{\text{atm}}\eta\Sigma_3$$

The atmospheric drag component of the right sides is calculated using the formulas

$$\dot{L}_{\text{atm}} = \varepsilon_{\text{atm}}L(\Sigma_1 + e\Sigma_2/\eta_2), \qquad \dot{\lambda}_{\text{atm}} = -2e\varepsilon_{\text{atm}}\Sigma_3\eta_{22}$$

$$\dot{h}_{\text{atm}} = Q_e\cos\varphi + Q_g\sin\varphi, \qquad \dot{k}_{\text{atm}} = Q_e\sin\varphi - Q_g\cos\varphi$$

Taking into account atmospheric perturbations, the right sides are

$$\dot{\lambda} = \dot{\lambda} + \dot{\lambda}_{\text{atm}}, \quad \dot{L} = \dot{L}_{\text{atm}}, \quad \dot{\theta} = 0, \quad \dot{h} = \dot{h} + \dot{h}_{\text{atm}}, \quad \dot{k} = \dot{k} + \dot{k}_{\text{atm}}$$

For the case $P_{\text{cr}} = 1$, we add

$$\dot{\lambda} = \dot{\lambda} + \dot{\lambda}_{\text{cr}}, \quad \dot{\theta} = \dot{\theta}_{\text{cr}}, \quad \dot{h} = \dot{h} + \dot{h}_{\text{cr}}, \quad \dot{k} = \dot{k} + \dot{k}_{\text{cr}}, \quad \dot{\Omega} = \dot{\Omega} + \dot{\Omega}_{\text{cr}}$$

Thus, we finish determination of the right sides of the differential equations in the mode $P_{\text{tp}} = 2$. Further, under $P_{\text{tp}} = 1$, the short-period periodics, periodics from tesseral harmonics, osculating elements, position, and velocity are calculated.

Solve Kepler's equation and compute the short-period periodics using the equations described in Appendix G. The results of those calculations provide ΔL, Δe, $e\Delta g$, $\Delta\lambda$, $\Delta\Omega$, and Δi. The periodics from tesseral harmonics are calculated as described in Appendix K. The results of those calculations provide δL, δe, $e\delta g$, $\delta\lambda$, $\delta\Omega$, and δi. Total increments of the elements are calculated as follows:

$$\Delta L = \Delta L - \delta L, \qquad \Delta e = \Delta e' + \Delta e - \delta e$$
$$\Delta i = \Delta i' + \Delta i - \delta i, \qquad \Delta\Omega = \Delta\Omega' + \Delta\Omega - \delta\Omega$$
$$e\Delta g = e\Delta g' + e\Delta g - e\delta g, \qquad \Delta\lambda = \Delta\lambda' + \Delta\lambda - \delta\lambda$$
$$\Delta h = \Delta e\cos\varphi + e\Delta g\sin\varphi, \qquad \Delta k = \Delta e\sin\varphi - e\Delta g\cos\varphi$$

Osculating elements are calculated according to

$$L_n = L + \Delta L, \qquad \theta_n = \theta - s\Delta i, \qquad \lambda_n = \lambda + \Delta\lambda$$
$$\Omega_n = \Omega + \Delta\Omega, \qquad h_n = h + \Delta h, \qquad k_n = k + \Delta k$$

The angles λ_n and Ω_n are brought to the $[0, 2\pi)$ interval. Finally, position and velocity are calculated using the equations in Appendix L.

SB03: Module for Choosing the Integration Step

The step for integration of the averaged differential equations is set to a multiple of the satellite's period and is approximately equal to 1 day.

SB04: Module of Bessel's Function Calculation

Bessel's functions $I_0(z)$, $I_1(z)$, $I_2(z)$, $I_3(z)$, and $I_4(z)$ are calculated according to the following formulas.

If $z > z^*$, then

$$I_0(z) = I_1(z) = I_2(z) = I_3(z) = I_4(z) = 1/\sqrt{2\pi z}, \qquad A = A - z$$

where $z^* = 20$.

If $z \le z^*$, then

$$I_3(z) = \sum_0^\infty b_i$$

where

$$b_i = b_{i-1}\left(\frac{z}{2}\right)^2 \Big/ (i(i+3)), \qquad i = 1, 2, \ldots, b_0 = \left(\frac{z}{2}\right)^3 \Big/ 6$$

$$I_4(z) = \sum_0^\infty a_i$$

where

$$a_i = a_{i-1}\left(\frac{z}{2}\right)^2 \Big/ (i(i+4)), \qquad i = 1, 2, \ldots, a_0 = \left(\frac{z}{2}\right)^4 \Big/ 24$$

$$I_2(z) = I_4(z) + 6\frac{I_3(z)}{z}, \qquad I_1(z) = I_3(z) + 4\frac{I_2(z)}{z}$$
$$I_0(z) = I_2(z) + 2\frac{I_1(z)}{z}$$

Algorithm Description

First, the critical inclination flag is determined by

$$P_{\text{cr}} = \begin{cases} 1, & \text{if } |1 - 5\theta_{\text{in}}^2| < C_\theta \\ 0, & \text{if } |1 - 5\theta_{\text{in}}^2| \ge C_\theta \end{cases}$$

where $C_\theta = 0.15$.

The input element set λ_{in}, L_{in}, θ_{in}, Ω_{in}, h_{in}, and k_{in} and epoch t_{in} are transformed to operative elements λ, L, θ, Ω, h, k, and t.

Then, by cyclic access to SB01 and SB03, prediction of the mean elements for time t_n is fulfilled (provided $P_f = 1$). Osculating elements and position are attained by access to SB02 with $P_{\text{tp}} = 1$.

For lifetime calculations ($P_f = 1$), integration of the equations by means of the SB01 with a 1-day step is carried out until the failure of the SB01 module. Then integration proceeds with a 1-revolution step size up to the failure of the SB01. The resulting point is considered to be the time of decay and is the output of the procedure.

Appendix D: Secular Gravity Terms

Preliminary quantities are calculated:

$$e_2 = h^2 + k^2, \qquad e = \sqrt{e_2}, \qquad \eta_2 = 1 - e_2, \qquad \eta_4 = \eta_2^2$$
$$\eta = \sqrt{\eta_2}, \qquad \eta_1 = \frac{1}{\eta}, \qquad \eta_{22} = \frac{1}{1+\eta}, \qquad G = L\eta$$
$$\tilde{G} = \frac{G}{\sqrt{\mu R_e}}, \qquad \tilde{L} = \frac{L}{\sqrt{\mu R_e}}, \qquad L_2 = \frac{1}{L}, \qquad \tilde{L}_1 = \frac{1}{\tilde{L}}$$
$$\tilde{L}_2 = \tilde{L}_1^3, \qquad B_{20} = 1 - \theta^2, \qquad si = \sqrt{B_{20}}, \qquad \kappa = 1 - 5\theta^2$$
$$B_g = 3 - 5\theta^2, \qquad B_a = 1 - 3\theta^2, \qquad B_i = \theta \cdot si, \qquad s_1 = \frac{1}{si}$$
$$\xi = \frac{e}{1+\eta}, \qquad \xi_6 = \frac{\eta_2}{1+\eta}, \qquad \xi_7 = \xi\eta_2, \qquad \xi_{13} = \theta \cdot s_1$$
$$\xi_8 = e\eta_1^2\xi_{13}, \qquad n = \frac{\mu^2}{L^3}, \qquad \gamma_n = \frac{n}{n_e}, \qquad \gamma = \gamma_n 10^{-30}$$

The flag P_{cr} indicating critical inclination is defined as

$$P_{\text{cr}} = \begin{cases} 1, & |\kappa| < c_\kappa \\ 0, & |\kappa| \ge c_\kappa \end{cases}$$

where $c_\kappa = 0.15$. Next, compute

$$g_0 = \tan^{-1}(h, k), \qquad \sin\varphi_0 = \frac{k}{e}, \qquad \cos\varphi_0 = \frac{h}{e}$$
$$l_0 = \lambda - g_0, \qquad \tilde{\kappa}_1 = \frac{\mu}{R_e}\frac{\tilde{L}_2}{G}, \qquad \tilde{j}_2 = 0.25\frac{C_{20}}{\tilde{G}^4}, \qquad \kappa_2 = \frac{k_0}{\tilde{G}^7}$$

where $C_{20} = JM[1]$ is the second gravitational zonal harmonic of the Earth and $k_0 = \frac{3}{128}C_{20}^2$.

Coefficients of the secular evolution from the zonal harmonics up to sixth order for the A model and up to the eighth order for the AP model are calculated.

$$A_1 = B_a, \qquad dA_1 = -6\theta, \qquad A_2 = 3 - 30\theta^2 + 35\theta^4$$
$$dA_2 = \theta(-60 + 140\theta^2), \qquad A_3 = 5 - 105\theta^2 + 315\theta^4 - 213\theta^6$$
$$dA_3 = \theta(-210 + 1260\theta^2 - 1386\theta^4)$$

$$b_1 = 1 - 2\theta^2 - 7\theta^4, \qquad db_1 = -4\theta(1 + 7\theta^2), \qquad b_2 = B_a^2$$
$$db_2 = -12\theta B_a, \qquad b_3 = 5 - 18\theta^2 + 5\theta^4, \qquad db_3 = \theta(-36 + 20\theta^2)$$

$$M_1 = 1, \qquad dM_1 = 0, \qquad M_2 = 5 - 3\eta_2, \qquad dM_2 = -6\eta$$
$$M_3 = 63 - 70\eta_2 + 15\eta_4, \qquad dM_3 = \eta(-140 + 60\eta_2)$$

$$s_5 = -5b_1 + 4\eta b_2 + \eta_2 b_3, \qquad s_6 = 4b_2 + 2\eta b_3$$
$$s_7 = -5db_1 + 4\eta db_2 + \eta_2 db_3$$

For the AP and NA models:

$$A_4 = 35 - 1260\theta^2 + 6930\theta^4 - 12{,}012\theta^6 + 6435\theta^8$$
$$dA_4 = \theta(-2520 + 27{,}720\theta^2 - 72{,}072\theta^4 + 51{,}480\theta^6)$$

$$M_4 = 429 - 693\eta_2 + 315\eta_4 - 35\eta_4\eta_2$$
$$dM_4 = \eta(-1386 + 1260\eta_2 - 210\eta_4)$$

For the A model, $i = 1, \ldots, 3$; for the AP and NA models, $i = 1, \ldots, 4$:

$$s_{1i} = j_i M_i A_i, \quad s_{2i} = (4i-1)s_{1i}, \quad s_{3i} = j_i M_i dA_i, \quad s_{4i} = j_i dM_i A_i$$

where

$$j_i = \bar{\alpha}_i C_{2i0}/\tilde{G}^{4i-1}, \qquad \bar{\alpha}_1 = -1/4, \qquad \bar{\alpha}_2 = -3/128$$
$$\bar{\alpha}_3 = -5/2048, \qquad \bar{\alpha}_4 = -35/262,144 \text{ constants}$$

C_{2i0} are the coefficients of even zonal harmonics, retrieved from the JM array:

$$C_{20} = JM[1], \qquad C_{40} = JM[3], \qquad C_{60} = JM[5], \qquad C_{80} = JM[7]$$

For the A model, $i = 1, \ldots, 3$; for the AP and NA models, $i = 1, \ldots, 4$,

$$\Sigma_1 = \sum_{i=1}^{i1} s_{1i}, \qquad \Sigma_2 = \sum_{i=1}^{i1} s_{2i}, \qquad \Sigma_3 = \sum_{i=1}^{i1} s_{3i}$$
$$\Sigma_4 = \sum_{i=1}^{i1} s_{4i}, \qquad \xi_1 = \eta(\Sigma_4 + \kappa_2 s_6)$$
$$\bar{\beta} = \eta\tilde{\kappa}_1(3(\Sigma_1 + \kappa_2 s_5) + \xi_1), \qquad \dot{l} = n + \bar{\beta}$$
$$\dot{\Omega} = -\tilde{\kappa}_1(\Sigma_3 + \kappa_2 s_7), \qquad \dot{g} = \tilde{\kappa}_1(\Sigma_2 + 7\kappa_2 s_5 - \xi_1) - \theta\dot{\Omega}$$

Appendix E: A and AP Atmospheric Models

Atmospheric-drag coefficients are calculated according to the formulas

$$a_0 = L^2/\mu, \qquad h_n = a_0(1-e) - R_e(1 - \alpha_{er}B_{20}\cos^2\varphi_0)$$

where α_{er} is the Earth-oblateness coefficient. If $h_n > 1500$, we take $h_n = 1500$. If $h_n < 120$, then $\rho_0 = \rho_{\max}$ and $H = 6$, where

$$\rho_{\max} \simeq 2.5 \times 10^{-9}\ \text{kg/m}^3 \simeq 250.0\ (\text{kg}\cdot\text{s}^2)/\text{km}^4$$

and H is the scale height.

If $120 \le h_n \le 1500$, we calculate the number of a layer, defining the subarray of atmosphere parameters:

$$N_l = \begin{cases} 0, & 120 \le h_n \le 180 \\ 1, & 180 < h_n \le 320 \\ 2, & 320 < h_n \le 600 \\ 3, & 600 < h_n \le 1500 \end{cases}$$

$$z = \tilde{a}_2[N_l]\sqrt{h_n - \tilde{a}_3[N_l]} - \tilde{a}_4[N_l], \qquad \rho_0 = \tilde{a}_1[N_l]P(z)$$
$$H = 2\sqrt{h_n - \tilde{a}_3[N_l]}/\tilde{a}_2[N_l]$$

where $\tilde{a}_i$ $(i = 1, \ldots, 4)$ are constants from the atmosphere parameters array, corresponding to the fixed level of solar activity F_0, and $P(z)$ is the polynomial fit for the function e^z:

$$P(z) = (1 + z(u_1 + z(u_2 + z(u_3 + zu_4))))^8$$

where u_1, u_2, u_3, and u_4 are constants given in Appendix H. We are talking about four atmospheric layers in this Appendix, as we have introduced an additional layer for parameters $\tilde{a}_i$ for simplification of the approximation of the exponent. For all other parameters we are using three layers, as described in Appendix J, and as can be seen from the tables in Appendix H:

$$\xi_2 = \tilde{j}_2\frac{7-e}{1+e}, \qquad z_3 = \frac{a_0 e}{H}$$

$$Q = \begin{cases} \dfrac{\rho_0\mu}{\eta\sqrt{2\pi z_3}} & \text{if } z_3 > 3 \\ \dfrac{\rho_0\mu}{\eta}e^{-z_3}\left(1 + \frac{1}{4}z_3^2 + \frac{1}{4}\left(\frac{1}{4}z_3^2\right)^2\right) & \text{if } z_3 \le 3 \end{cases}$$

For further calculations, if $z_3 > 1$, we take

$$z_3 = 1, \qquad \xi_3 = \frac{3SQ}{L}, \qquad \ddot{l} = \xi_3 n, \qquad \ddot{g} = -\frac{\xi_2\ddot{l}\kappa}{2}, \qquad \ddot{\Omega} = \xi_2\ddot{l}\theta$$
$$\dot{L} = -\frac{\xi_3 L}{3}, \qquad \dot{e} = -\frac{2}{3}(1-e)\xi_3, \qquad x = \frac{2}{3}\frac{a_0}{H}\xi_3$$

Appendix F: Long-Period Terms for A, AP, and NA Algorithms

Long-period terms are obtained by differentiating Eq. (F1) for the defining function:

$$\mathbf{S}^* = \mathbf{S}^*_B - \frac{8}{3(1-5\theta^2)}\sum_{n=3}^{6}\frac{c_{n0}a_0^{n-2}\mu^{n-2}}{G^{2n-5}c_{20}}\sum_{k=1}^{n-1}\frac{M^k_{n-1}(e)P^k_n(0)A^k_n(i)\sin^k i}{2^k k!k} \tag{F1}$$

The following recurrent formula is used in [6] for the inclination functions:

$$A^k_n(i) = \frac{2n-1}{n+k}A^k_{n-1}(i) - \frac{n-k-1}{n+k}A^k_{n-2}(i)$$

with the initial conditions $A^k_k(i) = 1$ and $A^k_{k+1}(i) = \cos(i)$. As shown by an arrow in Fig. F1, the counting is done for every k in the direction of increasing n, beginning with $n = k+2$. However, $P^k_n(0) = 0$ for $(n-k)$ odd, and half of the values found in every column are not needed. This is why we have developed a recurrent scheme, in which we first recurrently calculate values in the $A1_3$–$A1_6$ diagonal, then calculate two intermediate values in the $A2_4$–$A2_5$ diagonal and then the needed values in the $A3_5$–$A3_6$ diagonal. After that, explicit formulas are employed to calculate values in the diagonal of $A4_7$–$A4_8$. Thus, we calculate only two unneeded values: $A2_4$ and $A2_5$. The figure shows the diagonal sequence by increasing the shading of the cells, and also by the first symbol used to denote the cell. For example, $A4_7$ means that the cell is located in the fourth calculated diagonal and in the seventh line of the table.

A similar technique is employed for calculating the eccentricity functions $M^k_{n-1}(e)$. Note that an eccentricity function has $\mathcal{O}(e^k)$, and therefore our algorithm does not take into the account the terms with $k > 4$, and the two top terms $A3_7$ and $A3_8$ are also neglected. To avoid the singularity at $e \to 0$, we calculate functions $M^k_{n-1}(e)e^{-1}$.

	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8
n=1								
n=2								
n=3	$A1_3$							
n=4	$A2_4$	$A1_4$						
n=5	$A3_5$	$A2_5$	$A1_5$					
n=6		$A3_6$	$A2_6$	$A1_6$				
n=7	$A4_7$		$A3_7$	$A2_7$				
n=8		$A4_8$		$A3_8$				

Fig. F1 Recurrent coefficient calculation.

For the sake of simplicity, let us denote functions $M_{n-1}^k(e)e^{-1}$, corresponding to $A_n^k(i)$, by $m1_n$, $m3_n$, and $m4_n$. Calculation of perturbations of the highly eccentric orbits (as well as the resonance orbits) requires more time and special algorithms, which are described in [10].

The inclination and eccentricity functions needed to obtain the coefficients of the long-period terms are calculated using the recursive formulas:

$$c_0 = en_0 = 1, \qquad c_i = sc_{i-1}, \qquad en_i = \frac{e}{2} en_{i-1}, \qquad i = 1, \ldots, 4$$

In the cycle for i extending from 3 to 6, the following coefficients are computed:

$$m1_i = (i-1)en_{i-3}/2, \qquad dm1_i = (i-2)m1_i$$

$$\text{if } i \geq 5, \quad \text{then } m3_i = (i-2)m1_i + \frac{(i-1)(i-2)}{6} m1_{i-2}$$

$$dm3_i = (i-4)m3_i + 2dm1_i$$

$$\text{if } i = 3, \quad \text{then } a1_i = -\kappa/4, \qquad da1_i = 5\theta/2$$

$$\text{if } i > 3, \quad \text{then } a1_i = a1_{i-1} + \delta_i, \qquad da1_i = da1_{i-1} - \Delta_i$$

where

$$\delta_i = \frac{s^2}{2(i-1)(i-2)}, \qquad \Delta_i = \frac{\theta}{(i-1)(i-2)}$$

$$\text{if } i = 4, \quad \text{then } a2_i = a1_{i-1} - s^2/2, \qquad da2_i = da1_i + \theta$$

$$\text{if } i > 4, \quad \text{then } a2_i = a2_{i-1} + 3\delta_{i-1}$$

$$da2_i = da2_{i-1} - 3\Delta_{i-1} \qquad a3_i = a2_i a2_{i-1} - 3\delta_{i-1}$$

$$da3_i = a2_{i-1}da2_i + a2_i da2_{i-1} + 3\Delta_{i-1}$$

For the AP or NA models, also compute terms for the seventh and eighth zonal harmonics:

$$m4_7 = 3 + 30en_3(1 + en_3)$$

$$dm4_7 = 3 + 30en_3(3 + 5en_3)$$

$$m4_8 = en_2(10.5 + en_3(70 + 52.5en_3))$$

$$dm4_7 = en_2(21 + en_3(280 + 315en_3))$$

$$a4_7 = -35 + 945\theta^2 - 3465\theta^4 + 3003\theta^6$$

$$da4_7 = \theta(1890 - 13860\theta^2 + 18018\theta^4)$$

$$a4_8 = -1 + 33\theta^2 - 143\theta^4 + 143\theta^6$$

$$da4_8 = \theta(66 - 572\theta^2 + 858\theta^4)$$

Preliminary quantities are computed:

$$B_e = B_{20}(1 - 15\theta^2), \qquad B_2 = 2\theta(8 - 15\theta^2), \qquad \nu = \theta/B_{20}$$

If $P_{\text{cr}} = 0$, we take $\xi_4 = 1/\kappa G$, $\xi_5 = 10\theta/\kappa$, and $\tilde{C}_i = C_{i0}$, where $i = 2, \ldots, 8$ for the AP and NA algorithms and $i = 2, \ldots, 6$ for the A algorithm. $\tilde{C}_{i0}$ constants are given in Appendix H.

If $P_{\text{cr}} = 1$, we take $\xi_4 = \tilde{L}_2/G\tilde{G}^4$ and $\tilde{C}_i = \approx{C}_{i0}$, where $i = 2, \ldots, 8$ for the AP and NA algorithms and $i = 2, \ldots, 6$ for the A algorithm. $\approx{C}_{i0}$ constants are given in Appendix H. Now calculate

$$j_y = 3n\tilde{j}_2\eta_1^2 e, \qquad j_{eg} = j_y e(15\theta^2 - 2), \qquad j_\Omega = -5j_y\theta$$

$$j_\lambda = j_y((15\theta^2 - 2) - 1.5\eta\kappa)$$

Then in the cycle for j extending from 1 to 4, calculate the coefficients S_{ij} ($i = 1, \ldots, 4$):

$$S_{1j} = \tilde{C}_{j+2}P1_j c_j/\tilde{G}^{2j-1}, \qquad dA = da1_{j+2} - j\nu a1_{j+2}$$

$$S_{2j} = S_{1j}m1_{j+2}dA, \qquad S_{3j} = S_{1j}dm1_{j+2}a1_{j+2}$$

$$S_{1j} = S_{1j}m1_{j+2}a1_{j+2}, \qquad S_{4j} = S_{1j}n_5$$

where

$$n_5 = \begin{cases} 2j - 1, & P_{\text{cr}} = 0 \\ 2j + 3 & P_{\text{cr}} = 1 \end{cases}$$

and $P1_j$ constants are given in Appendix H.

If $j \leq 2$, then calculate coefficient S_{ij}^2 ($i = 1, \ldots, 4$ and $j = 1, 2$):

$$S_{1j}^2 = \tilde{C}_{j+4}P2_j c_j/\tilde{G}^{2j+3}, \qquad dA = da3_{j+4} - j\nu \cdot a3_{j+4}$$

$$S_{2j}^2 = S_{1j}^2 m3_{j+4}dA, \qquad S_{3j}^2 = S_{1j}^2 a3_{j+4}dm3_{j+4}$$

$$S_{1j}^2 = S_{1j}^2 a3_{j+4}m3_{j+4}, \qquad S_{4j}^2 = S_{1j}^2(n_5 + 4)$$

where $P2_j$ constants are given in Appendix H. We take $S_{ij} = S_{ij} + S_{ij}^2$, where $i = 1, \ldots, 4$ and $j = 1, 2$.

For the AP or NA models, add the seventh and eighth zonal harmonic terms:

$$S_{1j}^3 = \tilde{C}_{j+6}P3_j c_j/\tilde{G}^{2j+7}, \qquad dA = da4_{6+j} - j\nu a4_{6+j}$$

$$S_{2j}^3 = S_{1j}^3 m4_{6+j}dA, \qquad S_{3j}^3 = S_{1j}^3 a4_{6+j}dm4_{6+j}$$

$$S_{1j}^3 = S_{1j}^3 a4_{6+j}m4_{6+j}, \qquad S_{4j}^3 = S_{1j}^3(n_5 + 8)$$

where $P3_j$ ($j = 1, 2$) constants are given in Appendix H. We take $S_{ij} = (S_{ij} + S_{ij}^2 + S_{ij}^3)$, where $i = 1, \ldots, 4$ and $j = 1, 2$.

Add the long-period periodic-term coefficients from the second zonal harmonic:

$$S_{12} = S_{12} + S_e/2, \qquad S_{22} = S_{22} - e\tilde{C}_2 B_2/\tilde{G}^3$$

$$S_{32} = S_{32} + S_e, \qquad S_{42} = S_{42} + 3S_e/2$$

where $S_e = \tilde{C}_2 e B_e/\tilde{G}^3$.

If $P_{\text{cr}} = 0$, then calculate the long-period periodic coefficients A_{ij} ($i = 1, \ldots, 4$ and $j = 1, \ldots, 4$) using the formulas

$$A_{1j} = -\eta_2\xi_4 j S_{1j}, \qquad A_{2j} = e\xi_4(\xi_5 S_{1j} + S_{2j})$$

$$A_{3j} = -\xi_4(e_2 S_{4j} + \eta_2 S_{3j}) - e\theta A_{2j}$$

$$A_{4j} = -e\xi_4(S_{4j} + \xi_6 S_{3j}) - \theta A_{2j}$$

If $P_{\text{cr}} = 1$, then calculate the long-period periodic coefficients A_{ij} ($i = 1, \ldots, 4$ and $j = 1, \ldots, 4$) using the formulas

$$S_{42} = S_{42} + 2S_e, \qquad A_{1j} = \eta_2\xi_4 j S_{1j}, \qquad A_{2j} = e\xi_4 S_{2j}$$

$$A_{3j} = -\xi_4(e_2 S_{4j} + \eta_2 S_{3j}) - e\theta A_{2j}$$

$$A_{4j} = -e\xi_4(3\eta S_{1j} + S_{4j} + \xi_6 S_{3j}) - \theta A_{2j}$$

Appendix G: Short-Period Periodics

Solve Kepler's equation for $\sin E''$ and $\cos E''$ (using an iterative technique),

$$l'' = E'' - e'' \sin E''$$

with initial approximation

$$E_0'' = l'' + e'' \sin l'' + \frac{e''^2}{2} \sin 2l'', \qquad E_{i+1}'' = E_i'' - \Delta E_i$$

$$\Delta E_i = \frac{E_i'' - l'' - e'' \sin E_i''}{1 - e'' \cos E_i''}$$

Iterations continue until $|\Delta E_i| < 10^{-7}$.

Then calculate preliminary quantities needed for the short-period periodics:

$$\left(\frac{a}{r}\right) = \frac{1}{1 - e'' \cos E''}, \qquad \cos f'' = \left(\frac{a}{r}\right)(\cos E'' - e'')$$

$$\sin f'' = \eta'' \left(\frac{a}{r}\right) \sin E''$$

Using addition formulas, compute the sines and cosines of the angles:

$$2g'' + f'', \qquad 2g'' + 2f''$$

$$2g'' + 3f''(\sin g'' = \cos\varphi, \cos g'' = \sin\varphi)$$

$$\alpha_2 = \left(\frac{a}{r}\right)^2 \eta_4 + \left(\frac{a}{r}\right)\eta_2 + 1, \qquad \alpha_3 = \left(\frac{a}{r}\right)^2 \eta_2 + \left(\frac{a}{r}\right) + 1$$

$$\alpha_4 = 2 - \alpha_3, \qquad \alpha_5 = \alpha_3 - 2/3, \qquad f'' = \tan^{-1}(\sin f'', \cos f'')$$

$$\bar{\beta} = f'' - l'' + e'' \sin f'', \qquad \alpha_6 = e'' + \alpha_2 \cos f''$$

$$\alpha_7 = \alpha_6 + \xi_7$$

$$\alpha_8 = 3(\sin(2g'' + 2f'') + e'' \sin(2g'' + f'')) + e'' \sin(2g'' + 3f'')$$

$$\alpha_9 = 3\cos(2g'' + f'') + \cos(2g'' + 3f'')$$

Short-period periodics are computed in the following sequence:

$$\Delta L = \tilde{j}_2 L'' \left(-B_a e'' \eta_1^2 \alpha_7 + 3B_{20}\eta_4 \left(\frac{a}{r}\right)^3 \cos(2g'' + 2f'') \right)$$

$$\Delta e = \tilde{j}_2(-B_a\alpha_7 + B_{20}(3\alpha_6 \cos(2g'' + 2f'') - \eta_2\alpha_9))$$

$$\Delta\lambda = \tilde{j}_2(-3\kappa\bar{\beta} + 0.5B_g\alpha_8)$$

$$e\Delta g = \tilde{j}_2\eta_2(-B_a\alpha_3 \sin f'' + 1.5B_{20}(\alpha_4 \sin(2g'' + f'') + \alpha_5 \sin(2g'' + 3f''))) + e''\Delta\lambda$$

$$\Delta\lambda = \Delta\lambda + \xi(e\Delta g - e''\Delta\lambda), \qquad \Delta\Omega = \tilde{j}_2\theta(\alpha_8 - 6\bar{\beta})$$

$$\Delta i = \tilde{j}_2 B_i(3\cos(2g'' + 2f'') + e''\alpha_9)$$

Appendix H: Constants

The values of the most frequently used physical and mathematical constants are given in this section: $\mu = 398,600.44$ is the gravitational constant, $n_e = 0.7292115855\text{E} - 4$ is the Earth rotation velocity, $R_e = 6378.137$ is the Earth radius, $c_\kappa = 0.15$ is the critical inclination test limit, $C_{20} = JM[1] = J_2$ is the second gravitational zonal harmonic of the Earth, $\alpha_{\text{er}} = 1/298.256$ is the Earth-oblateness coefficient, and JM is the array of the zonal geopotential harmonics:

$$JM = \{0.108262741\text{E} - 2, \ -0.253444\text{E} - 5, \ -0.162483\text{E} - 5, \ -0.229288\text{E} - 6, \ 0.543970\text{E} - 6, \ -0.359881\text{E} - 6, \ -0.210113\text{E} - 6\}$$

$$\tilde{C}_{20} = C_{20}\sqrt{\mu R_e}/16, \qquad \tilde{C}_{i0} = \frac{8}{3}\frac{JM[i-1]}{C_{20}}\sqrt{\mu R_e} \qquad i = 3, \ldots, 8$$

$$\tilde{\tilde{C}}_{20} = \frac{3}{64}\frac{\mu}{R_e}C_{20}^2, \qquad \tilde{\tilde{C}}_{i0} = 2\frac{\mu}{R_e}JM[i-1] \qquad i = 3, \ldots, 8$$

$$P_1 = \{0, \ -0.75, \ -0.46875, \ -0.3645833333, \ -0.3076171875\}$$

$$P_2 = \{0, \ 0.9375, \ 0.8203125\} \qquad P_3 = \{0, \ -5/2^{11}, \ -315/2^{13}\}$$

The polynomial fitting coefficients for the exponential approximation are

$$u_1 = -0.9998684/8, \qquad u_2 = 0.4982926/8^2$$

$$u_3 = -0.1595332/8^3, \qquad u_4 = 0.0293641/8^4$$

As we calculate the density of the atmosphere, we are using the values for the coefficients of the model for the F_0 that is closest to the 135 day mean, $\bar{F}_{135}$.

The constants $\tilde{a}_i$ $(i = 1, \ldots, 4)$ of the layers of the atmospheric density model approximation are listed subsequently for fixed values of solar activity level. These are used in both the static and the dynamic atmospheric models described in Appendices E and I, respectively. An additional layer is introduced for these constants.

For $F_0 = 75$,

$$\tilde{a}_1 = \{2488.5533, 40.293687, 0.4496995, 0.001213834\}$$
$$\tilde{a}_2 = \{0.7009, 0.8016, 0.8016, 0.2336\}$$
$$\tilde{a}_3 = \{115.3429, 86.3329, 86.3329, 491.2201\}$$
$$\tilde{a}_4 = \{1.512564, 7.758026, 12.25339, 2.43639\}$$

For $F_0 = 100$,

$$\tilde{a}_1 = \{2488.3927, 43.839081, 0.6731885, 0.002518119\}$$
$$\tilde{a}_2 = \{0.0.7000, 0.7675, 0.7675, 0.2417\}$$
$$\tilde{a}_3 = \{114.638, 77.1052, 77.1052, 490.7284\}$$
$$\tilde{a}_4 = \{1.620829, 7.785295, 11.96155, 2.52656\}$$

For $F_0 = 125$,

$$\tilde{a}_1 = \{2488.4695, 53.128673, 1.0484604, 0.005169129\}$$
$$\tilde{a}_2 = \{0.6419, 0.7362, 0.7362, 0.2654\}$$
$$\tilde{a}_3 = \{115.956, 70.5386, 70.5386, 479.9537\}$$
$$\tilde{a}_4 = \{1.290698, 7.702404, 11.62780, 2.90787\}$$

For $F_0 = 150$,

$$\tilde{a}_1 = \{2488.3167, 60.041084, 1.4225368, 0.009801573\}$$
$$\tilde{a}_2 = \{0.6124, 0.6805, 0.6805, 0.2911\}$$
$$\tilde{a}_3 = \{116.415, 80.4406, 80.4406, 461.9691\}$$
$$\tilde{a}_4 = \{1.159459, 6.789992, 10.53258, 3.42003\}$$

For $F_0 = 200$,

$$\tilde{a}_1 = \{2487.8713, 67.399254, 2.1926321, 0.027742197\}$$
$$\tilde{a}_2 = \{0.5974, 0.5797, 0.5797, 0.3492\}$$
$$\tilde{a}_3 = \{116.214, 100.9417, 100.9417, 399.0605\}$$
$$\tilde{a}_4 = \{1.162338, 5.154387, 8.57992, 4.95002\}$$

For $F_0 = 250$,

$$\tilde{a}_1 = \{2488.1558, 75.038546, 3.0883709, 0.060895223\}$$
$$\tilde{a}_2 = \{0.5772, 0.5095, 0.5095, 0.4130\}$$
$$\tilde{a}_3 = \{116.339, 115.2277, 115.2277, 284.6955\}$$
$$\tilde{a}_4 = \{1.104324, 4.100519, 7.29088, 7.35132\}$$

Additional constants of the layers of the atmospheric density model approximation are listed subsequently for fixed values of solar activity level. All of the following constants have three layers. These are used in the dynamic atmospheric model described in Appendix I.

For $F_0 = 75$,

$$b_1 = \{-6.828000\text{e} - 01, \ -8.607000\text{e} - 01, \ 7.833000\text{e} - 01\}$$
$$b_2 = \{5.576200\text{e} - 03, \ 7.861000\text{e} - 03, \ 2.861000\text{e} - 03\}$$
$$b_3 = \{9.523800\text{e} - 07, \ -5.711000\text{e} - 06, \ -1.944000\text{e} - 06\}$$

$c_1 = \{-4.384000e+00,\ 1.279100e+00,\ -4.400000e+00\}$
$c_2 = \{8.063000e-02,\ -1.576000e-02,\ 3.024000e-02\}$
$c_3 = \{-4.925000e-04,\ 6.499000e-05,\ -3.283000e-05\}$
$c_4 = \{1.042000e-06,\ -5.145000e-08,\ 1.012000e-08\}$
$d_1 = \{-5.101900e+00,\ -1.721000e-01,\ 1.020400e+00\}$
$d_2 = \{6.258100e-02,\ 5.756000e-03,\ 2.499000e-03\}$
$d_3 = \{-1.672100e-04,\ -3.635000e-06,\ -1.519000e-06\}$
$e_1 = \{-7.238000e+00,\ -2.152000e-01,\ -3.800000e+00\}$
$e_2 = \{1.203000e-01,\ 4.167000e-03,\ 1.972000e-02\}$
$e_3 = \{-6.450000e-04,\ 1.587000e-06,\ -1.833000e-05\}$
$e_4 = \{1.208000e-06,\ -1.651000e-09,\ 4.938000e-09\}$
$e_5 = \{-1.200000e-01,\ -1.200000e-01,\ -1.200000e-01\}$
$e_6 = \{5.000000e-03,\ 5.000000e-03,\ 5.000000e-03\}$
$e_7 = \{1.500000e-02,\ 1.500000e-02,\ 1.500000e-02\}$
$l_1 = \{-1.197500e-02,\ -1.698000e-02,\ 1.083000e-02\}$
$l_2 = \{9.983000e-05,\ 1.448000e-04,\ 6.694000e-05\}$
$l_3 = \{0.000000e+00,\ -9.535000e-08,\ -4.277000e-08\}$
$n = \{1.500000e+00,\ 1.500000e+00,\ 1.500000e+00\}$
$\varphi_1 = \{5.411000e-01,\ 5.411000e-01,\ 5.411000e-01\}$

For $F_0 = 100$,

$b_1 = \{-7.804000e-01,\ -7.540000e-01,\ 7.250000e-01\}$
$b_2 = \{7.173000e-03,\ 6.850000e-03,\ 2.675000e-03\}$
$b_3 = \{-5.578000e-06,\ -4.600000e-06,\ -1.750000e-06\}$
$c_1 = \{-4.384000e+00,\ 1.279100e+00,\ -4.400000e+00\}$
$c_2 = \{8.063000e-02,\ -1.576000e-02,\ 3.024000e-02\}$
$c_3 = \{-4.925000e-04,\ 6.499000e-05,\ -3.283000e-05\}$
$c_4 = \{1.042000e-06,\ -5.145000e-08,\ 1.012000e-08\}$
$d_1 = \{-5.101900e+00,\ -1.721000e-01,\ 1.020400e+00\}$
$d_2 = \{6.258000e-02,\ 5.756000e-03,\ 2.499000e-03\}$
$d_3 = \{-1.672200e-04,\ -3.635000e-06,\ -1.519000e-06\}$
$e_1 = \{-6.683000e+00,\ -2.162000e-01,\ -3.700000e+00\}$
$e_2 = \{1.101400e-01,\ 4.086000e-03,\ 1.783000e-02\}$
$e_3 = \{-5.837500e-04,\ 1.270000e-06,\ -1.506000e-05\}$
$e_4 = \{1.083000e-06,\ -1.587000e-09,\ 3.580000e-09\}$
$e_5 = \{-1.200000e-01,\ -1.200000e-01,\ -1.200000e-01\}$
$e_6 = \{2.500000e-02,\ 2.500000e-02,\ 2.500000e-02\}$
$e_7 = \{7.500000e-03,\ 7.500000e-03,\ 7.500000e-03\}$
$l_1 = \{-9.900000e-03,\ -1.249000e-02,\ 8.317000e-03\}$
$l_2 = \{8.212000e-05,\ 1.111000e-04,\ 4.837000e-05\}$
$l_3 = \{3.125000e-09,\ -7.706000e-08,\ -3.039000e-08\}$
$n = \{1.500000e+00,\ 1.500000e+00,\ 1.500000e+00\}$
$\varphi_1 = \{5.515000e-01,\ 5.515000e-01,\ 5.515000e-01\}$

For $F_0 = 125$,

$b_1 = \{-8.220000e-01,\ -5.700000e-01,\ 6.100000e-01\}$
$b_2 = \{8.330000e-03,\ 5.250000e-03,\ 2.343000e-03\}$
$b_3 = \{-1.233000e-05,\ -3.000000e-06,\ -1.433000e-06\}$
$c_1 = \{9.776000e-01,\ 1.290300e+00,\ -8.980000e+00\}$
$c_2 = \{-2.570000e-02,\ -1.547000e-02,\ 4.087000e-02\}$
$c_3 = \{2.027000e-04,\ 5.964000e-05,\ -3.950000e-05\}$
$c_4 = \{-4.708000e-07,\ -4.503000e-08,\ 1.123000e-08\}$
$d_1 = \{-5.101900e+00,\ -1.721000e-01,\ 1.020400e+00\}$
$d_2 = \{6.258000e-02,\ 5.756000e-03,\ 2.499000e-03\}$
$d_3 = \{-1.672200e-04,\ -3.635000e-06,\ -1.519000e-06\}$
$e_1 = \{-5.352000e+00,\ -1.486000e-01,\ -3.700000e+00\}$
$e_2 = \{8.615000e-02,\ 3.263000e-03,\ 1.750000e-02\}$
$e_3 = \{-4.437500e-04,\ 3.143000e-06,\ -1.500000e-05\}$
$e_4 = \{8.125000e-07,\ -3.429000e-09,\ 3.704000e-09\}$
$e_5 = \{-1.000000e-01,\ -1.000000e-01,\ -1.000000e-01\}$
$e_6 = \{2.083000e-02,\ 2.083000e-02,\ 2.083000e-02\}$
$e_7 = \{6.251000e-03,\ 6.250000e-03,\ 6.251000e-03\}$
$l_1 = \{-7.680000e-03,\ -7.879000e-03,\ 4.667000e-03\}$
$l_2 = \{6.362000e-05,\ 7.258000e-05,\ 4.606000e-05\}$
$l_3 = \{3.125000e-09,\ -3.658000e-08,\ -2.722000e-08\}$
$n = \{1.500000e+00,\ 1.500000e+00,\ 1.500000e+00\}$
$\varphi_1 = \{5.585000e-01,\ 5.585000e-01,\ 5.585000e-01\}$

For $F_0 = 150$,

$b_1 = \{-7.376000e-01,\ -4.760000e-01,\ 9.333000e-02\}$
$b_2 = \{7.597000e-03,\ 4.400000e-03,\ 3.038000e-03\}$
$b_3 = \{-1.209000e-05,\ -2.400000e-06,\ -1.711000e-06\}$
$c_1 = \{9.776000e-01,\ 1.290300e+00,\ -8.980000e+00\}$
$c_2 = \{-2.570000e-02,\ -1.547000e-02,\ 4.087000e-02\}$
$c_3 = \{2.027000e-04,\ 5.964000e-05,\ -3.950000e-05\}$
$c_4 = \{-4.708000e-07,\ -4.503000e-08,\ 1.123000e-08\}$
$d_1 = \{-5.101900e+00,\ -1.721000e-01,\ 1.020400e+00\}$
$d_2 = \{6.258000e-02,\ 5.756000e-03,\ 2.499000e-03\}$
$d_3 = \{-1.672200e-04,\ -3.635000e-06,\ -1.519000e-06\}$
$e_1 = \{-4.799000e+00,\ -1.495000e-01,\ -4.400000e+00\}$
$e_2 = \{7.779200e-02,\ 3.182000e-03,\ 1.981000e-02\}$
$e_3 = \{-4.075000e-04,\ 2.825000e-06,\ -1.806000e-05\}$
$e_4 = \{7.708300e-07,\ -3.365000e-09,\ 4.938000e-09\}$
$e_5 = \{-1.000000e-01,\ -1.000000e-01,\ -1.000000e-01\}$
$e_6 = \{2.750000e-02,\ 2.750000e-02,\ 2.750000e-02\}$
$e_7 = \{3.800000e-03,\ 3.750000e-03,\ 3.800000e-03\}$
$l_1 = \{-5.600000e-03,\ -4.882000e-03,\ -1.333000e-02\}$
$l_2 = \{4.667000e-05,\ 4.692000e-05,\ 7.167000e-05\}$
$l_3 = \{0.000000e+00,\ -1.742000e-08,\ -3.518000e-08\}$
$n = \{1.500000e+00,\ 1.500000e+00,\ 1.500000e+00\}$
$\varphi_1 = \{5.585000e-01,\ 5.585000e-01,\ 5.585000e-01\}$

For $F_0 = 200$,

$b_1 = \{-5.161000e-01,\ -3.113000e-01,\ -4.333000e-01\}$

$b_2 = \{5.341000e-03,\ 2.839000e-03,\ 3.522000e-03\}$

$b_3 = \{-8.672000e-06,\ -1.089000e-06,\ -1.889000e-06\}$

$c_1 = \{-5.632000e-01,\ 2.057000e-01,\ -1.578000e+01\}$

$c_2 = \{5.743000e-03,\ -2.911000e-03,\ 5.757000e-02\}$

$c_3 = \{-9.250000e-06,\ 1.739000e-05,\ -5.322000e-05\}$

$c_4 = \{4.167000e-09,\ -8.565000e-09,\ 1.512000e-08\}$

$d_1 = \{-5.101900e+00,\ -1.721000e-01,\ 1.020400e+00\}$

$d_2 = \{6.258000e-02,\ 5.756000e-03,\ 2.499000e-03\}$

$d_3 = \{-1.672200e-04,\ -3.635000e-06,\ -1.519000e-06\}$

$e_1 = \{-5.115000e+00,\ -8.286000e-02,\ -3.600000e+00\}$

$e_2 = \{8.507500e-02,\ 2.278000e-03,\ 1.653000e-02\}$

$e_3 = \{-4.587500e-04,\ 4.381000e-06,\ -1.528000e-05\}$

$e_4 = \{8.750000e-07,\ -5.143000e-09,\ 4.321000e-09\}$

$e_5 = \{-1.100000e-01,\ -1.100000e-01,\ -1.100000e-01\}$

$e_6 = \{3.810000e-02,\ 3.810000e-02,\ 3.810000e-02\}$

$e_7 = \{1.178000e-03,\ 1.178000e-03,\ 1.178000e-03\}$

$l_1 = \{-4.110000e-03,\ -5.017000e-03,\ -1.500000e-03\}$

$l_2 = \{3.463000e-05,\ 4.282000e-05,\ 3.250000e-05\}$

$l_3 = \{-3.125000e-09,\ -2.132000e-08,\ -1.389000e-08\}$

$n = \{1.500000e+00,\ 1.500000e+00,\ 1.500000e+00\}$

$\varphi_1 = \{5.585000e-01,\ 5.585000e-01,\ 5.585000e-01\}$

For $F_0 = 250$,

$b_1 = \{-2.531000e-01,\ -3.307000e-01,\ -1.750000e-01\}$

$b_2 = \{1.929000e-03,\ 2.878000e-03,\ 2.642000e-03\}$

$b_3 = \{1.495000e-06,\ -1.378000e-06,\ -1.417000e-06\}$

$c_1 = \{4.842000e-01,\ 1.499000e-03,\ -9.750000e+00\}$

$c_2 = \{-1.604000e-02,\ -2.399000e-04,\ 3.383000e-02\}$

$c_3 = \{1.405000e-04,\ 7.006000e-06,\ -2.694000e-05\}$

$c_4 = \{-3.375000e-07,\ -5.999000e-10,\ 6.481000e-09\}$

$d_1 = \{-5.101900e+00,\ -1.721000e-01,\ 1.020400e+00\}$

$d_2 = \{6.258000e-02,\ 5.756000e-03,\ 2.499000e-03\}$

$d_3 = \{-1.672200e-04,\ -3.635000e-06,\ -1.519000e-06\}$

$e_1 = \{-3.137000e+00,\ -2.048000e-01,\ 1.000000e-01\}$

$e_2 = \{4.774200e-02,\ 3.596000e-03,\ 2.639000e-03\}$

$e_3 = \{-2.275000e-04,\ -1.587000e-06,\ -2.778000e-07\}$

$e_4 = \{3.958300e-07,\ 3.175000e-10,\ -6.173000e-10\}$

$e_5 = \{-9.000000e-02,\ -9.000000e-02,\ -9.000000e-02\}$

$e_6 = \{3.118000e-02,\ 3.117000e-02,\ 3.118000e-02\}$

$e_7 = \{9.662000e-04,\ 9.662000e-04,\ 9.662000e-04\}$

$l_1 = \{-3.030000e-03,\ -5.455000e-03,\ -8.333000e-04\}$

$l_2 = \{2.532000e-05,\ 4.273000e-05,\ 2.817000e-05\}$

$l_3 = \{-5.556000e-10,\ -2.273000e-08,\ -1.129000e-08\}$

$n = \{1.500000e+00,\ 1.500000e+00,\ 1.500000e+00\}$

$\varphi_1 = \{5.585000e-01,\ 5.585000e-01,\ 5.585000e-01\}$

Coefficients are listed in Table H2 for the semiannual effect on density. The days are measured from the beginning of the year.

Appendix I: Dynamic Atmosphere Model

The atmosphere subroutine provides an approximation of the atmospheric density model with the constants given in Appendix H. The input parameters are as follows: h is the satellite altitude above the Earth surface, $\sin u$ and $\cos u$ are the sine and cosine of the argument of latitude (under $P_{\mathrm{fl}} = 0$), $\sin i$ and $\cos i$ are the sine and cosine of inclination, Ω is the longitude of the ascending node, t is the epoch time, and P_{fl} is the flag of the operation mode.

The subroutine uses two operation modes: $P_{\mathrm{fl}} = 0$ is the calculation of density for the required point, and $P_{\mathrm{fl}} = 1$ is the calculation of the mean density and parameters of the solar bulge. The flag $P_{\mathrm{fl}} = 0$ is used for the numerical integration of the equations of motion. Access by the NA procedure to the subroutine is done with $P_{\mathrm{fl}} = 1$.

The density ρ, the mean density $\bar{\rho}_0$, and the solar bulge parameters F, $\cos u^*$, and $\sin u^*$ are calculated using the formulas

$$\rho = \rho_n K_0 K_1 K_2 K_3 K_4$$

$$\bar{\rho}_0 = \rho_n K_0 K_2 K_3 K_4 (1 + 0.5 K_1'(U_1 + U_2))$$

$$F = \frac{K_1'(U_1 - U_2)}{2 + K_1'(U_1 + U_2)}, \qquad \cos u^* = \kappa_1/\kappa, \qquad \sin u^* = \kappa_2/\kappa$$

where

$$K_0 = 1 + (l_1 + l_2 h + l_3 h^2) \cdot (\bar{F}_{135} - F_0) \cdot K_f$$

$$K_f = 10^{-22}\ (\mathrm{m^2 Hz})/\mathrm{W}, \qquad K_2 = 1 + (d_1 + d_2 h + d_3 h^2) A(D)$$

$$K_3 = 1 + (b_1 + b_2 h + b_3 h^2)(\bar{F} - \bar{F}_{135})/\bar{F}_{135}$$

$$K_4 = 1 + (e_1 + e_2 h + e_3 h^2 + e_4 h^3)(e_5 + e_6 \bar{k}_p + e_7 \bar{k}_p^2)$$

are multipliers, characterizing changes in the atmospheric density due to the F_0 deviation from $\bar{F}_{135}$, the half-year effect, the $\bar{F}$ deviation

Table H1 Harmonics *C* and *S*

C_{22} 157.303E − 8	S_{22} −900.645E − 9
C_{31} 2.195707352324075e − 006	S_{31} 2.689831426874183e − 007
C_{32} 3.097516590539374e − 007	S_{32} −2.115278046617040e − 007
C_{33} 9.984003539968880e − 008	S_{33} 1.953238609278902e − 007
C_{41} −5.040733835861600e − 007	S_{41} −4.477310825149399e − 007
C_{42} 7.827132348440263e − 008	S_{42} 1.457156058217513e − 007
C_{43} 5.912078793211645e − 008	S_{43} −1.181304196035648e − 008
C_{44} −4.083151493157146e − 009	S_{44} 6.312457128567290e − 009
C_{51} −5.768365776659221e − 008	S_{51} −7.241285779012821e − 008
C_{52} 1.055065265370098e − 007	S_{52} −5.280343203131510e − 008
C_{53} −1.491336797624337e − 008	S_{53} −6.642882193693165e − 009
C_{54} −2.282858417546628e − 009	S_{54} 4.234177180196651e − 010
C_{55} 4.716907103893786e − 010	S_{55} −1.648763029132619e − 009
C_{61} −6.110256124317695e − 008	S_{61} 2.534269247544076e − 008
C_{62} 4.729814906274774e − 009	S_{62} −4.538109362598566e − 008
C_{63} 1.174574848502832e − 009	S_{63} 5.473746866809312e − 011
C_{64} −3.440616429237547e − 010	S_{64} −1.760850630678994e − 009
C_{65} −2.168665866081844e − 010	S_{65} −4.317881452169429e − 010
C_{66} 3.487706994310528e − 012	S_{66} −5.472694542174636e − 011
C_{71} 2.049902500730218e − 007	S_{71} 6.342862524128992e − 008
C_{72} 3.198232553810615e − 008	S_{72} 9.648482948851894e − 009
C_{73} 3.411324290169202e − 009	S_{73} −3.450060526844259e − 009
C_{74} −5.763472676170248e − 010	S_{74} −2.586671849542382e − 010
C_{75} 6.229021228682090e − 013	S_{75} 5.432697492060800e − 012
C_{76} −2.483272562861421e − 011	S_{76} 1.045727267017083e − 011
C_{77} 2.637886150381004e − 013	S_{77} 6.051184685332514e − 013
C_{81} 1.618318958054932e − 008	S_{81} 4.021402353519092e − 008
C_{82} 5.728041203740248e − 009	S_{82} 4.328473923675238e − 009
C_{83} −1.963366823420749e − 010	S_{83} −9.155638492738570e − 010
C_{84} −3.171893623539545e − 010	S_{84} 8.643025090559981e − 011
C_{85} −3.925857011115952e − 012	S_{85} 1.462829707876400e − 011
C_{86} −1.854184511452173e − 012	S_{86} 8.652861053443472e − 012
C_{87} 3.496421663537376e − 013	S_{87} 3.760042780651102e − 013
C_{88} −1.449151285952052e − 013	S_{88} 1.614998033227177e − 013

Table H2 Semiannual variation coefficients

D, days	$A(D)$	D, days	$A(D)$	D, days	$A(D)$
0	−0.028	130	0.013	260	0.015
10	−0.045	140	−0.037	270	0.070
20	−0.047	150	−0.086	280	0.115
30	−0.035	160	−0.128	290	0.144
40	−0.011	170	−0.162	300	0.155
50	0.022	180	−0.185	310	0.145
60	0.057	190	−0.199	320	0.120
70	0.090	200	−0.202	330	0.084
80	0.114	210	−0.193	340	0.044
90	0.125	220	−0.173	350	0.006
100	0.118	230	−0.140	360	−0.023
110	0.096	240	−0.096	370	−0.040
120	0.060	250	−0.042	——	——

from $\bar{F}_{135}$, and the dependence of atmospheric density on geomagnetic perturbation, respectively; $\bar{F}$, F_0, and $\bar{F}_{135}$ are the daily averaged, fixed, and mean for the preceding 135 days' values of $F_{10.7}$; $F_{10.7}$ is the index of solar activity, equal to the solar radio-emission flux density for the 10.7 cm wavelength; $\bar{k}_p$ is the daily averaged index of geomagnetic perturbations; D is the amount of days from the beginning of the year; $\rho_n = \tilde{a}_1 P(z)$ is the approximation of the atmosphere night density; $z = \tilde{a}_2 - \tilde{a}_3\sqrt{h - \tilde{a}_4}$; and $P(z)$ is the polynomial approximation of e^z:

$$P(z) = (1 + z(u_1 + z(u_2 + z(u_3 + zu_4))))^8$$

where u_1, u_2, u_3, and u_4 are constants given in Appendix H, and $K_1 = 1 + K_1' \cos^n\varphi/2$ is the factor accounting for the daily effect in density distribution:

$$K_1' = c_1 + c_2 h + c_3 h^2 + c_4 h^3$$

where φ is the central angle between the current point at which the density is calculated and the point with the maximum density regarding its daily distribution:

$$\cos\varphi = \kappa_1 \cos u + \kappa_2 \sin u$$

$$\kappa_1 = \sin i \sin\delta_\odot - \sin(\Omega - \alpha_\odot - \varphi_1)\cos i \cos\delta_\odot$$

$$\kappa_2 = \cos(\Omega - \alpha_\odot - \varphi_1)\cos\delta_\odot$$

where $\alpha_\odot$ and $\delta_\odot$ are the angular coordinates of the sun,

$$\kappa = \sqrt{\kappa_1^2 + \kappa_2^2}, \qquad U_1 = \left(\tfrac{1}{2}(1+\kappa)\right)^{n/2}, \qquad U_2 = \left(\tfrac{1}{2}(1-\kappa)\right)^{n/2}$$

$K_f, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, b_1, b_2, b_3, l_1, l_2, l_3, c_1, c_2, c_3, c_4, d_1, d_2, d_3, e_1, e_2, e_3, e_4, e_5, e_6, e_7, n$, and φ_1, are the coefficients of the model, stored in the MKA array and used for calculating the atmospheric density under different values of F_0 and three layers: $120\ \text{km} \le h \le 180$ km, $180\ \text{km} < h \le 600$ km, and $600\ \text{km} < h \le 1500$ km. These are listed in Appendix H.

For night density calculations,

$$\rho_n = a_1 \exp(a_2 - a_3\sqrt{h - a_4})$$

Constants $\tilde{a}_i$ ($i = 1, \ldots, 4$) used for its polynomial fit, are specified for not three, as in [7], but for four layers: $120\ \text{km} \le h \le 180$ km, $180\ \text{km} < h \le 320$ km, $320\ \text{km} < h \le 600$ km, and $600\ \text{km} < h \le 1500$ km.

Their values are constant and are given in Appendix H.

Appendix J: Tesseral Harmonic Coefficients

The AP and NA models calculate periodics from tesseral harmonics up to the eighth order. The perturbation equations are based on Kaula [4].

The BAX1 subroutine calculates the coefficients of the periodics of the tesseral harmonic perturbations, which are combinations of inclination and eccentricity functions and their derivatives. All of the variables are computed using explicit formulas. For eccentricity functions, only the main terms of the series in eccentricity are retained. That is why the accuracy of the calculations of the coefficients of the perturbations significantly decreases with the increase of the eccentricity (beyond 0.1). For resonance cases, the respective perturbations are not accounted for.

Using the following expression for the generating function,

$$\mathbf{S}_{nqkp} = \Re e\left\{\sqrt{-1}\,\frac{\mathcal{H}_{nqkp}}{(qn_e - pn_s)}\right\}$$

one can obtain equations for calculating the tesseral perturbations:

$$\left.\begin{aligned}
\Delta L &= \tfrac{\partial \mathbf{S}}{\partial l},\\
\Delta e &= \tfrac{\eta}{eL}\left(-\tfrac{\partial \mathbf{S}}{\partial g} + \eta\,\tfrac{\partial \mathbf{S}}{\partial l}\right),\\
\Delta i &= \tfrac{s_1}{G}\left(\theta\tfrac{\partial \mathbf{S}}{\partial g} - \tfrac{\partial \mathbf{S}}{\partial h}\right),\\
\Delta\Omega &= -\tfrac{s_1}{G}\tfrac{\partial \mathbf{S}}{\partial i},\\
e\Delta g &= \tfrac{\eta}{L}\tfrac{\partial \mathbf{S}}{\partial e} - \theta e \Delta\Omega,\\
\Delta\lambda &= -\tfrac{\partial \mathbf{S}}{\partial L} + \tfrac{\eta e}{(1+\eta)L}\tfrac{\partial \mathbf{S}}{\partial e} - \theta\Delta\Omega
\end{aligned}\right\} \tag{J1}$$

where n_e is the Earth rotation rate, n_S is the satellite mean motion, $\eta = \sqrt{1 - e^2}$, and

$$s_1 = \frac{1}{\sin(i)} = \frac{1}{2s \cdot c}$$

To avoid excessive arithmetic calculations, and taking into account that most of the formulas have either L or G in the denominator, we will use the reduced generating function, for which the real part in its complete form can be written as

$$\mathbf{S}_{nqkp} = \tilde{\gamma}_{nqp}\frac{1}{q\tilde{L}^{2n}} A^k_{nq} X_{nkp}\begin{cases}(-1)^{\frac{n-q-1}{2}} D^1_{nqkp} & (n-q)\ \text{odd}\\ (-1)^{\frac{n-q}{2}} D^2_{nqkp} & (n-q)\ \text{even}\end{cases}$$

where

$$D^1_{nqkp} = C_{nq}\cos\psi_{kqp} + S_{nq}\sin\psi_{kqp}$$

$$D^2_{nqkp} = -S_{nq}\cos\psi_{kqp} + C_{nq}\sin\psi_{kqp}$$

$$\psi_{kqp} = kg + q(\Omega - S_{\text{star}}) + pl, \qquad \tilde{\gamma}_{nqp} = \frac{n_S}{n_e}\left(1 - \frac{p}{q}\frac{n_S}{n_e}\right)^{-1}$$

In the preceding formulas, we used the following notation: S_{star} is the star time (Greenwich hour angle), and $\tilde{L} = L/\sqrt{\mu R_e}$. All of the necessary functions $A^k_{nq}(i)$ are calculated as

$$A^k_{nq} = s^{|q-k|}c^{|q+k|}P^k_{nq}(\theta), \qquad s = \sin\left(\frac{i}{2}\right), \qquad c = \cos\left(\frac{i}{2}\right)$$

and are given in Table J1. The formula for the generating function for the inclination is

$$\frac{d}{di}A^k_{nq} = \frac{s_2^{q-k}c_2^{q+k}}{\sin i}\left\{(q\theta - k)P^k_{nq}(\theta) - (1-\theta^2)\frac{d}{d\theta}P^k_{nq}(\theta)\right\}$$

and the resultant expressions are given in Table J2. The function for eccentricity is X_{nkp} and the resultant expressions are given in Table J3. The further actions consist of multiplications and additions, but to conserve computational resources, we use sophisticated tricks, keeping in memory parts of expressions and common factors. Using the following expression for the function,

$$e_i \equiv \{L, e, i, \omega, \Omega, l\}, \qquad \frac{\partial \mathbf{S}_{nqkp}}{\partial e_i} \equiv \mathbf{S}^{(e_i)}_{nqkp}$$

To avoid excessive arithmetic calculations, and taking into account that most of the formulas have either L or G in the denominator, we will use the reduced generating function, for which the real part in its complete form can be written as

$$\mathbf{S}_{nqkp} = \tilde{\gamma}_{nqp}\frac{L\tilde{L}^{2n}}{q}A^k_{nq}X_{nkp}\begin{cases}(-1)^{\frac{n-q-1}{2}}(C_{nq}\cos\psi_{nqkp} + S_{nq}\sin\psi_{nqkp}) & (n-q)\text{ odd}\\ (-1)^{\frac{n-q}{2}}(-S_{nq}\cos\psi_{nqkp} + C_{nq}\sin\psi_{nqkp}) & (n-q)\text{ even}\end{cases}$$

$$= L\frac{n_s}{n_e}\left(1-\frac{pn_s}{qn_e}\right)^{-1}(C\tilde{S}_{nqkp}\cos\psi_{nqkp} + S\tilde{S}_{nqkp}\sin\psi_{nqkp})$$

Define the following variables:

$$(\tilde{L})^{2n} = (\sqrt{\mu R_e}/L)^{2n}, \qquad \beta_{nq} = (-1)^m\tilde{L}^{2n}/q$$

$$m = \begin{cases}\dfrac{n-q}{2} & (n-q)\text{ even}\\ \dfrac{n-q-1}{2} & (n-q)\text{ odd}\end{cases}$$

$$\tilde{\mathbf{S}}_{nqkp} = \beta_{nq}A^k_{nq}X_{nkp}, \qquad C\tilde{S}_{nqkp} = \begin{cases}\tilde{S}_{nqkp}C_{nq} & (n-q)\text{ odd}\\ -\tilde{S}_{nqkp}S_{nq} & (n-q)\text{ even}\end{cases}$$

$$S\tilde{S}_{nqkp} = \begin{cases}\tilde{S}_{nqkp}S_{nq} & (n-q)\text{ odd}\\ \tilde{S}_{nqkp}C_{nq} & (n-q)\text{ even}\end{cases}$$

Table J1 Tesseral initialization [$s = \sin(\frac{i}{2}), c = \cos(\frac{i}{2}), \theta = \cos(i), s\cdot c = 0.5\sin(i), s^2 = 0.5(1-\theta)$, and $c^2 = 0.5(1+\theta)$]

n	q	k	$A^k_{n,q}$
4	1	0	$(-5.625\theta + 13.125\theta^3)cs$
6	1	0	$(8.203125\theta - 49.21875\theta^3 + 54.140625\theta^5)cs$
8	1	0	$(-10.7666\theta + 118.433\theta^3 - 307.925\theta^5 + 219.946\theta^7)cs$
2	2	0	$6(cs)^2$
4	2	0	$(-11.25 + 78.75\theta^2)(cs)^2$
6	2	0	$(16.4062 - 295.312\theta^2 + 541.406\theta^4)(cs)^2$
8	2	0	$(-21.5332 + 710.596\theta^2 - 3079.25\theta^4 + 3079.25\theta^6)(cs)^2$
4	3	0	$(315\theta)(cs)^3$
6	3	0	$(-1181.25\theta + 4331.25\theta^3)(cs)^3$
8	3	0	$(2842.38\theta - 24,634\theta^3 + 36,951\theta^5)(cs)^3$
4	4	0	$630(cs)^4$
6	4	0	$(-2362.5 + 25,987.5\theta^2)(cs)^4$
8	4	0	$(5684.77 - 147,804\theta^2 + 369,510\theta^4)(cs)^4$
6	5	0	$103,950\theta(\mathrm{cs})^5$
8	5	0	$(-591,216\theta + 2,956,078\theta^3)(cs)^5$
6	6	0	$207,900(cs)^6$
8	6	0	$(-1,182,431 + 17,736,469\theta^2)(cs)^6$
8	7	0	$(70,945,875\theta)(cs)^7$
8	8	0	$141,891,750(cs)^8$
3	1	1	$(-0.375 - 3.75\theta + 5.625\theta^2)c^2$
3	1	−1	$(-0.375 + 3.75\theta + 5.625\theta^2)s^2$
5	1	1	$(0.234375 + 6.5625\theta - 9.84375\theta^2 - 19.6875\theta^3 + 24.6094\theta^4)c^2$
5	1	−1	$(0.234375 - 6.5625\theta - 9.84375\theta^2 + 19.6875\theta^3 + 24.6094\theta^4)s^2$
7	1	1	$(-0.170898 - 9.22852\theta + 13.8428\theta^2 + 67.6758\theta^3 - 84.5947\theta^4 - 87.9785\theta^5 + 102.6418\theta^6)c^2$
7	1	−1	$(-0.170898 + 9.22852\theta + 13.8428\theta^2 - 67.6758\theta^3 - 84.5947\theta^4 + 87.9785\theta^5 + 102.6418\theta^6)s^2$
3	2	1	$(-7.5 + 22.5\theta)c^3s$
3	2	−1	$(7.5 + 22.5\theta)cs^3$
5	2	1	$(13.125 - 39.375\theta - 118.125\theta^2 + 196.875\theta^3)c^3s$
5	2	−1	$(-13.125 - 39.375\theta + 118.125\theta^2 + 196.875\theta^3)cs^3$
7	2	1	$(-18.457 + 55.3711\theta + 406.055\theta^2 - 676.758\theta^3 - 879.785\theta^4 + 1231.7\theta^5)c^3s$
7	2	−1	$(18.457 + 55.3711\theta - 406.055\theta^2 - 676.758\theta^3 + 879.785\theta^4 + 1231.7\theta^5)cs^3$
3	3	1	$45c^4s^2$
3	3	−1	$45c^2s^4$
5	3	1	$(-78.75 - 472.5\theta + 1181.25\theta^2)c^4s^2$
5	3	−1	$(-78.75 + 472.5\theta + 1181.25\theta^2)c^2s^4$
7	3	1	$(110.742 + 1624.22\theta - 4060.55\theta^2 - 7038.28\theta^3 + 12,317\theta^4)c^4s^2$
7	3	−1	$(110.742 - 1624.22\theta - 4060.55\theta^2 + 7038.28\theta^3 + 12,317\theta^4)c^2s^4$
5	4	1	$(-945 + 4725\theta)c^5s^3$
5	4	−1	$(945 + 4725\theta)c^3s^5$
7	4	1	$(3248.44 - 16,242.2\theta - 42,229.7\theta^2 + 98,535.9\theta^3)c^5s^3$
7	4	−1	$(-3248.44 - 16,242.2\theta + 42,229.7\theta^2 + 98,535.9\theta^3)c^3s^5$
5	5	1	$9450c^6s^4$
5	5	−1	$9450c^4s^6$
7	5	1	$(-32,484.4 - 168,919\theta + 591,216\theta^2)c^6s^4$
7	5	−1	$(-32,484.4 + 168,919\theta + 591,216\theta^2)c^4s^6$
7	6	1	$(-337,838 + 2,364,863\theta)c^7s^5$
7	6	−1	$(337,838 + 2,364,863\theta)c^5s^7$
7	7	1	$4,729,725c^8s^6$
7	7	−1	$4,729,725c^6s^8$

Table J2 Tesseral initialization

n	q	k	$(d/di)A_{nq}^{k} = A_{nq}^{k(i)}$
4	1	0	$(5.625 - 50.625\theta^2 + 52.5\theta^4)s_1$
6	1	0	$(-8.20312 + 164.0625\theta^2 - 467.578125\theta^4 + 324.84375\theta^6)s_1$
8	1	0	$(10.77 - 376.83\theta^2 + 2013.35\theta^4 - 3387.17\theta^6 + 1759.57\theta^8)s_1$
2	2	0	$12\theta s_1$
4	2	0	$(-180\theta + 315\theta^3)s_1$
6	2	0	$(623.4\theta - 3346.9\theta^3 + 3248.4\theta^5)s_1$
8	2	0	$(-1464\theta + 15,159\theta^3 - 36,951\theta^5 + 24,634\theta^7)s_1$
4	3	0	$(-315 + 1260\theta^2)s_1$
6	3	0	$(1181.25 - 17,719\theta^2 + 25,988\theta^4)s_1$
8	3	0	$(-2842.38 + 85,271\theta^2 - 332,559\theta^4 + 295,608\theta^6)s_1$
4	4	0	$2520\theta s_1$
6	4	0	$(-61,425\theta + 155,925\theta^3)s_1$
8	4	0	$(318,347\theta - 2,364,863\theta^3 + 2,956,078\theta^5)s_1$
6	5	0	$(-103,950 + 623,700\theta^2)s_1$
8	5	0	$(591,216 - 12,415,528\theta^2 + 23,648,625\theta^4)s_1$
6	6	0	$1,247,400\theta s_1$
8	6	0	$(-42,567,526\theta + 141,891,752\theta^3)s_1$
8	7	0	$(-70,945,875 + 567,567,000\theta^2)s_1$
8	8	0	$1,135,134,000\theta s_1$
3	1	1	$(4.125 - 7.875\theta - 13.125\theta^2 + 16.875\theta^3)s_1$
3	1	−1	$(-4.125 - 7.875\theta + 13.125\theta^2 + 16.875\theta^3)s_1$
5	1	1	$(-6.7969 + 13.3594\theta + 82.0313\theta^2 - 108.2813\theta^3 - 103.3594\theta^4 + 123.0469\theta^5)s_1$
5	1	−1	$(6.7969 + 13.3594\theta - 82.0313\theta^2 - 108.2813\theta^3 + 103.3594\theta^4 + 123.0469\theta^5)s_1$
7	1	1	$(9.3994 - 18.62790\theta - 235.3271\theta^2 + 312.2314\theta^3 + 795.1904\theta^4 - 950.8447\theta^5 - 630.5129\theta^6 + 718.4912\theta^7)s_1$
7	1	−1	$(-9.3994 - 18.6279\theta + 235.3271\theta^2 + 312.2314\theta^3 - 795.1904\theta^4 - 950.8447\theta^5 + 630.5129\theta^6 + 718.4912\theta^7)s_1$
3	2	1	$(-15 - 37.5\theta + 67.5\theta^2)s_1$
3	2	−1	$(-15 + 37.5\theta + 67.5\theta^2)s_1$
5	2	1	$(26.25 + 301.875\theta - 590.625\theta^2 - 669.375\theta^3 + 984.375\theta^4)s_1$
5	2	−1	$(26.25 - 301.875\theta - 590.625\theta^2 + 669.375\theta^3 + 984.375\theta^4)s_1$
7	2	1	$(-36.91 - 904.39\theta + 1790.33\theta^2 + 5820.12\theta^3 - 8662.5\theta^4 - 6510.41\theta^5 + 8621.89\theta^6)s_1$
7	2	−1	$(-36.91 + 904.39\theta + 1790.33\theta^2 - 5820.12\theta^3 - 8662.5\theta^4 + 6510.41\theta^5 + 8621.89\theta^6)s_1$
3	3	1	$(-45 + 135\theta)s_1$
3	3	−1	$(45 + 135\theta)s_1$
5	3	1	$(551.25 - 2126.25\theta - 3071.25\theta^2 + 5906.25\theta^3)s_1$
5	3	−1	$(-551.25 - 2126.25\theta + 3071.25\theta^2 + 5906.25\theta^3)s_1$
7	3	1	$(-1734.96 + 6829.1\theta + 31,672.27\theta^2 - 62,532.42\theta^3 - 54,546.68\theta^4 + 86,218.94\theta^5)s_1$
7	3	−1	$(1734.96 + 6829.1\theta - 31,672.27\theta^2 - 62,532.42\theta^3 + 54,546.68\theta^4 + 86,218.94\theta^5)s_1$
5	4	1	$(-3780 - 8505\theta + 23,625\theta^2)s_1$
5	4	−1	$(-3780 + 8505\theta + 23,625\theta^2)s_1$
7	4	1	$(12,994 + 113,695\theta - 334,589\theta^2 - 351,914\theta^3 + 689,752\theta^4)s_1$
7	4	−1	$(12,994 - 113,695\theta - 334,589\theta^2 + 351,914\theta^3 + 689,752\theta^4)s_1$
5	5	1	$(-9450 + 47,250\theta)s_1$
5	5	−1	$(9450 + 47,250\theta)s_1$
7	5	1	$(201,403 - 1,175,934\theta - 1,604,728\theta^2 + 4,138,509\theta^3)s_1$
7	5	−1	$(-201,403 - 1,175,934\theta + 1,604,728\theta^2 + 4,138,509\theta^3)s_1$
7	6	1	$(-2,027,025 - 4,393,888\theta + 16,554,038\theta^2)s_1$
7	6	−1	$(-2,027,025 + 4,393,888\theta + 16,554,038\theta^2)s_1$
7	7	1	$(-4,729,725 + 33,108,075\theta)s_1$
7	7	−1	$(4,729,725 + 33,108,075\theta)s_1$

Table J3 Function $X_{nkp}(n-k)$ even

Index		n						
k	p	2	3	4	5	6	7	8
0	0	$1 + 1.5e^2$	0	$1 + 5e^2$	0	$1 + 10.5e^2$	0	$1 + 18e^2$
0	±1	$1.5e$	0	0	0	0	0	0
±1	0	0	e	0	$2e$	0	$3e$	0
±1	±1	0	$1 + 2e^2$	0	0	0	0	0

Then the individual components $\tilde{\mathbf{S}}_{nqkp}$ needed for the perturbation equations are

$$\tilde{\mathbf{S}}_{nqkp}^{(L)} = \begin{cases} (2n+2)\tilde{\mathbf{S}}_{nqk0} \\ \left(2n + 2 + \frac{3pn_e}{(qn_e - pn_S)}\right)\tilde{\mathbf{S}}_{nqk(p=\pm1)} \end{cases}$$

$$\tilde{\mathbf{S}}_{nqk}^{(e)} = \beta_{nq} A_{nq}^{k} \frac{\partial X_{nkp}}{\partial e} \qquad \tilde{\mathbf{S}}_{nqkp}^{(i)} = \beta_{nq} \frac{\partial A_{nq}^{k}}{\partial i} X_{nkp}$$

To avoid excessive arithmetic calculations, and taking into account that most of the formulas have either e^0 or e^1, we will use the reduced function

$$C^1\tilde{\mathbf{S}}_{qkp}^{(e_i)}, \qquad S^1\tilde{\mathbf{S}}_{qkp}^{(e_i)}$$

for which its complete form can be written as

$$C\tilde{\mathbf{S}}_{qkp}^{(e_i)} = eC_{\Sigma}^{1}\tilde{\mathbf{S}}_{qkp}^{(e_i)}, \qquad S\tilde{\mathbf{S}}_{qkp}^{(e_i)} = eS_{\Sigma}^{1}\tilde{\mathbf{S}}_{qkp}^{(e_i)}$$

Given in Table J4 are all of the sets of indices n, q, k, and p used in the calculations. Some cells of the table give two indices: k in the first line and p in the second one. If $p = 0$, the second line is not given.

We can draw the following conclusions from analyzing this table:

1) The greatest number of terms is 19 and corresponds to the set $k = 0$ and $p = 0$.

Table J4 Indices n, q, k, and p

	q							
n	1	2	3	4	5	6	7	8
1	—	—	—	—	—	—	—	—
2	—	0	—	—	—	—	—	—
2	—	0, 1, −1	—	—	—	—	—	—
3	1 −1	1 −1	1 −1	—	—	—	—	—
3	1 −1	1 −1	1 −1	—	—	—	—	—
4	0	0	0	0	—	—	—	—
5	1 −1	1 −1	1 −1	1 −1	1 −1	—	—	—
6	0	0	0	0	0	0	—	—
7	1 −1	1 −1	1 −1	1 −1	1 −1	1 −1	1 −1	—
8	0	0	0	0	0	0	0	0

2) The set with $k = \pm 1$ and $p = 0$ contains 12 terms.
3) The set with $k = \pm 1$ and $p = \pm 1$ contains 3 terms.
4) There is one term with $k = 0$ and $p = \pm 1$.

The index structure has the following feature: there is no term with the general position $k \neq p \neq 0$. Taking these characteristics of the structure into account permits building an algorithm for calculating the perturbations with a relatively small number of multiplications, compared with an algorithm that calculates every term of each perturbation independently.

At this point, we will describe examples of economizing on the number of multiplications in calculating the amplitudes (and, in the next Appendix, in calculating the total perturbations).

The following values are calculated in sequence in advance and are kept in memory:

$$\theta^i = \theta^{i-1}\theta, \qquad c^i = c^{i-1}c, \qquad s^i = s^{i-1}s, \qquad c^i s^j = c^i s^j$$

$$(\tilde{L})^{2n} = (\sqrt{\mu a_e}/L)^{2n} = (\tilde{L})^{2n-2}(\tilde{L})^2$$

The following value is kept in memory:

$$b_{nq} = \begin{cases} (-1)^{\frac{n-q}{2}} & (n-q)\ \text{even} \\ (-1)^{\frac{n-q-1}{2}} & (n-q)\ \text{odd} \end{cases}$$

and the working cells are calculated:

$$wc1 = \begin{cases} C_{nq} & (n-q)\ \text{odd} \\ -S_{nq} & (n-q)\ \text{even} \end{cases}$$

$$ws1 = \begin{cases} S_{nq} & (n-q)\ \text{odd} \\ C_{nq} & (n-q)\ \text{even} \end{cases}$$

$$\beta_{nq} = \frac{b_{nq}\tilde{L}^{2n}}{q}$$

Amplitudes for $k = p = 0$

The first type of amplitude is calculated in a cycle in q (i.e., over the rows of the table), and the summation over n (the columns) is carried out inside this former cycle. Because $k = 0$, the derivative with respect to ω is equal to 0:

$$w1 = \beta_{nq} \cdot A_{nq}^{k=0}, \qquad w2 = \beta_{nq} \cdot A_{nq}^{k=0,(i)}, \qquad X_{n00}^{(e)} = e\tilde{X}_{n00}^{(e)}$$

$$\tilde{X}_{200}^{(e)} = 3e, \qquad \tilde{X}_{400}^{(e)} = 10e, \qquad \tilde{X}_{600}^{(e)} = 21e, \qquad \tilde{X}_{800}^{(e)} = 36e$$

$$C\tilde{S}_{nq}^{(L)} = (2n+2) \cdot w1 \cdot wc1, \qquad S\tilde{S}_{nq}^{(L)} = (2n+2) \cdot w1 \cdot ws1$$

$$C^1\tilde{S}_{nq}^{(e)} = w1 \cdot wc1 \cdot \tilde{X}_{n00}^{(e)}, \qquad S^1\tilde{S}_{nq}^{(e)} = w1 \cdot ws1 \cdot \tilde{X}_{n00}^{(e)}$$

$$C\tilde{S}_{nq}^{(i)} = w2 \cdot wc1, \qquad S\tilde{S}_{nq}^{(i)} = w2 \cdot ws1$$

$$C\tilde{S}_{nq}^{(\Omega)} = q \cdot w1 \cdot ws1, \qquad S\tilde{S}_{nq}^{(\Omega)} = -q \cdot w1 \cdot wc1$$

$$C_\Sigma\tilde{S}_{q,k=p=0}^{(L)} = \sum_{n=2,4,6,8} C\tilde{S}_{nq}^{(L)}, \qquad S_\Sigma\tilde{S}_{q,k=p=0}^{(L)} = \sum_{n=2,4,6,8} S\tilde{S}_{nq}^{(L)}$$

$$C_\Sigma^1\tilde{S}_{q,k=p=0}^{(e)} = \sum_{n=2,4,6,8} C^1\tilde{S}_{nq}^{(e)}, \qquad S_\Sigma^1\tilde{S}_{q,k=p=0}^{(e)} = \sum_{n=2,4,6,8} S^1\tilde{S}_{nq}^{(e)}$$

$$C_\Sigma\tilde{S}_{q,k=p=0}^{(i)} = \sum_{n=2,4,6,8} C\tilde{S}_{nq}^{(i)}, \qquad S_\Sigma\tilde{S}_{q,k=p=0}^{(i)} = \sum_{n=2,4,6,8} S\tilde{S}_{nq}^{(i)}$$

$$C_\Sigma\tilde{S}_{q,k=p=0}^{(\Omega)} = \sum_{n=2,4,6,8} C\tilde{S}_{nq}^{(\Omega)}, \qquad S_\Sigma\tilde{S}_{q,k=p=0}^{(\Omega)} \sum_{n=2,4,6,8} S\tilde{S}_{nq}^{(\Omega)}$$

$$C_\Sigma\tilde{S}_{q,k=p=0}^{(\omega)} = S_\Sigma\tilde{S}_{q,k=p=0}^{(\omega)} = 0$$

Amplitudes for $p = 0$ and $k = \pm 1$

Let us calculate the corresponding sums for $k = \pm 1$ and $p = 0$. In this case, the derivative of the Hansen coefficient for $n = 3, 5, 7$ will have values $\tilde{X}_{nk0}^{(e)} = 0.5(n-1) = [1, 2, 3]$. These coefficients will be employed in the formulas directly, without using the working cells. Thus, the Hansen coefficients (Table J4) will be e, $2e$, and $3e$ and will be calculated without the factor e to save on the multiplications:

$$w1_{nq}^{k=\pm 1} = \beta_{nq} \cdot A_{nq}^{k=\pm 1}, \qquad w2_{nq}^{k=\pm 1} = \beta_{nq} \cdot A_{nq}^{k=\pm 1,(i)}$$

$$X_{310}^{(e)} = 1, \qquad X_{510}^{(e)} = 2, \qquad X_{710}^{(e)} = 3$$

$$\tilde{X}_{n,|k|=1,p=0} = X_{n,|k|=1,p=0}^{(e)} = \tfrac{1}{e}X_{n,|k|=1,p=0}$$

$$C\tilde{S}_{nq,k=\pm 1,p=0}^{(e)} = wc1 \cdot X_{n,k=\pm 1,p=0}^{(e)} \cdot w1_{nq}^{k=\pm 1}$$

$$S\tilde{S}_{nq,k=\pm 1,p=0}^{(e)} = ws1 \cdot X_{n,k=\pm 1,p=0}^{(e)} \cdot w1_{nq}^{k=\pm 1}$$

$$C^1\tilde{S}_{nq,k=\pm 1,p=0}^{(L)} = (2n+2) \cdot C\tilde{S}_{nq,k=\pm 1}^{(e)}$$

$$S^1\tilde{S}_{nq,k=\pm 1,p=0}^{(L)} = (2n+2) \cdot S\tilde{S}_{nq,k=\pm 1}^{(e)}$$

$$C^1\tilde{S}_{nq,k=\pm 1,p=0}^{(i)} = wc1 \cdot 0.5(n-1) \cdot w2_{nq}^{k=\pm 1}$$

$$S^1\tilde{S}_{nq,k=\pm 1,p=0}^{(i)} = ws1 \cdot 0.5(n-1) \cdot w2_{nq}^{k=\pm 1}$$

$$C^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\Omega)} = q \cdot S^1\tilde{S}_{nq,k=\pm 1}^{(e)}$$

$$S^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\Omega)} = -q \cdot C^1\tilde{S}_{nq,k=\pm 1}^{(e)}$$

$$C^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\omega)} = k \cdot S^1\tilde{S}_{nq,k=\pm 1}^{(e)}$$

$$S^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\omega)} = -k \cdot C^1\tilde{S}_{nq,k=\pm 1}^{(e)}$$

$$C_\Sigma^1\tilde{S}_{q,k=\pm 1,p=0}^{L} = \sum_{n=3,5,7} C^1\tilde{S}_{nq,k=\pm 1,p=0}^{(L)}$$

$$S_\Sigma^1\tilde{S}_{q,k=\pm 1,p=0}^{L} = \sum_{n=3,5,7} S^1\tilde{S}_{nq,k=\pm 1,p=0}^{(L)}$$

$$C_\Sigma\tilde{S}_{q,k=\pm 1,p=0}^{(e)} = \sum_{n=3,5,7} C\tilde{S}_{nq,k=\pm 1,p=0}^{(e)}$$

$$S_\Sigma\tilde{S}_{q,k=\pm 1,p=0}^{(e)} = \sum_{n=3,5,7} S\tilde{S}_{nq,k=\pm 1,p=0}^{(e)}$$

$$C_\Sigma^1\tilde{S}_{nq,k=\pm 1,p=0}^{(i)} = \sum_{n=3,5,7} C^1\tilde{S}_{nq,k=\pm 1,p=0}^{(i)}$$

$$S_\Sigma^1\tilde{S}_{nq,k=\pm 1,p=0}^{(i)} = \sum_{n=3,5,7} S^1\tilde{S}_{nq,k=\pm 1,p=0}^{(i)}$$

$$C_\Sigma^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\Omega)} = \sum_{n=3,5,7} C^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\Omega)}$$

$$S_\Sigma^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\Omega)} = \sum_{n=3,5,7} S^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\Omega)}$$

$$C_\Sigma^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\omega)} = \sum_{n=3,5,7} C^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\omega)}$$

$$S_\Sigma^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\omega)} = \sum_{n=3,5,7} S^1\tilde{S}_{nq,k=\pm 1,p=0}^{(\omega)}$$

Amplitudes for $k = p = \pm 1$

Let us calculate the corresponding sums for the short-period perturbations $k = p = \pm 1$. In this case, the following short-period factors emerge:

$$\tilde{\gamma}_q = \left(1 - \frac{p}{q}\gamma\right)^{-1}, \qquad \gamma = \frac{n_s}{n_e} \equiv \frac{\mu^2/L^3}{n_e}$$

We will denote them for $p = 1$ and $p = -1$ as

$$\tilde{\gamma}_q^+ = \left(1 - \frac{1}{q}\gamma\right)^{-1}, \qquad \tilde{\gamma}_q^- = \left(1 + \frac{1}{q}\gamma\right)^{-1}$$

The following working cells are calculated:

$$\beta_{nq}^{p=\pm1} = b_{nq}\tilde{L}^{2n}\tilde{\gamma}_q^{p=\pm1}/q, \qquad w1_{nq}^{k=p\pm1} = \beta_{nq}^{p=\pm1} \cdot A_{nq}^{k=\pm1}$$
$$w2_{nq}^{k=p=\pm1} = \beta_{nq}^{p=\pm1} \cdot A_{nq}^{k=\pm1,(i)}$$
$$wc11_{nq}^{k=p=\pm1} = wc1 \cdot w1_{nq}^{k=p=\pm1}$$
$$ws11_{nq}^{k=p=\pm1} = ws1 \cdot w1_{nq}^{k=p\pm1}$$
$$C\tilde{S}_{nq,k=\pm1}^{(L)} = \left(2n + 2 + \frac{3\gamma}{q}\tilde{\gamma}_q^{p=\pm1}\right) \cdot wc11_{nq}^{k=p=\pm1}$$
$$S\tilde{S}_{nq,k=\pm1}^{(L)} = \left(2n + 2 + \frac{3\gamma}{q}\tilde{\gamma}_q^{p=\pm1}\right) \cdot ws11_{nq}^{k=p=\pm1}$$
$$C^1\tilde{S}_{nq,k=p=\pm1}^{(e)} = wc1 \cdot \tilde{X}_{n=3,k=p=\pm1}^{(e)} \cdot w1_{nq}^{k=p=\pm1}$$
$$S^1\tilde{S}_{nq,k=p=\pm1}^{(e)} = ws1 \cdot \tilde{X}_{n=3,k=p=\pm1}^{(e)} \cdot w1_{nq}^{k=p=\pm1}$$
$$C\tilde{S}_{nq,k=p=\pm1}^{(i)} = wc1 \cdot X_{n=3,k=p=\pm1} \cdot w2_{nq}^{k=p=\pm1}$$
$$S\tilde{S}_{nq,k=p=\pm1}^{(i)} = ws1 \cdot X_{n=3,k=p=\pm1} \cdot w2_{nq}^{k=p=\pm1}$$

where

$$X_{n=3,k=p=\pm1} = 1, \qquad X_{n=3,k=p=\pm1}^{(e)} = 4e, \qquad \tilde{X}_{n=3,k=p=\pm1}^{(e)} = 4$$
$$C\tilde{S}_{nq,k=p=\pm1}^{(\omega)} = k \cdot ws11_{nq}^{k=p=\pm1}$$
$$S\tilde{S}_{nq,k=p=\pm1}^{(\omega)} = -k \cdot wc11_{nq}^{k=p=\pm1}$$
$$C\tilde{S}_{nq,k=p=\pm1}^{(\Omega)} = q \cdot ws11_{nq}^{k=p=\pm1}$$
$$S\tilde{S}_{nq,k=p=\pm1}^{(\Omega)} = -q \cdot wc11_{nq}^{k=p=\pm1}$$

$$C_\Sigma\tilde{S}_{q,k=p=\pm1}^{(L)} = \sum_{n=3,q=1,2,3} C\tilde{S}_{nq,k=p=\pm1}^{(L)}$$
$$S_\Sigma\tilde{S}_{q,k=p=\pm1}^{(L)} = \sum_{n=3,q=1,2,3} S\tilde{S}_{nq,k=p=\pm1}^{(L)}$$
$$C_\Sigma^1\tilde{S}_{q,k=p=\pm1}^{(e)} = \sum_{n=3,q=1,2,3} C^1\tilde{S}_{nq,k=p=\pm1}^{(e)}$$
$$S_\Sigma^1\tilde{S}_{q,k=p=\pm1}^{(e)} = \sum_{n=3,q=1,2,3} S^1\tilde{S}_{nq,k=p=\pm1}^{(e)}$$
$$C_\Sigma\tilde{S}_{q,k=p=\pm1}^{(i)} = \sum_{n=3,q=1,2,3} C\tilde{S}_{nq,k=p=\pm1}^{(i)}$$
$$S_\Sigma\tilde{S}_{q,k=p=\pm1}^{(i)} = \sum_{n=3,q=1,2,3} S\tilde{S}_{nq,k=p=\pm1}^{(i)}$$
$$C_\Sigma\tilde{S}_{q,k=p=\pm1}^{(\Omega)} = \sum_{n=3,q=1,2,3} C\tilde{S}_{nq,k=p=\pm1}^{(\Omega)}$$
$$S_\Sigma\tilde{S}_{q,k=p=\pm1}^{(\Omega)} \sum_{n=3,q=1,2,3} S\tilde{S}_{nq,k=p=\pm1}^{(\Omega)}$$
$$C_\Sigma\tilde{S}_{q,k=p=\pm1}^{(\omega)} = \sum_{n=3,q=1,2,3} C\tilde{S}_{nq,k=p=\pm1}^{(\omega)}$$
$$S_\Sigma\tilde{S}_{q,k=p=\pm1}^{(\omega)} = \sum_{n=3,q=1,2,3} S\tilde{S}_{nq,k=p=\pm1}^{(\omega)}$$

Amplitudes for $p = \pm 1, k = 0, n = 2$, and $q = 2$

Finally, let us calculate perturbations from the second sectoral harmonics:

$$\tilde{\gamma}_2^+ = (1 - 0.5\gamma)^{-1}, \qquad \tilde{\gamma}_2^- = (1 + 0.5\gamma)^{-1}$$
$$\beta_{22}^{p=\pm1} = b_{nq}\tilde{L}^{2n}\tilde{\gamma}_2^{p=\pm1}/2, \qquad w1_{22}^{k=0,p=\pm1} = \beta_{22}^{p=\pm1} \cdot A_{22}^{k=0}$$
$$w2_{22}^{k=0,p=\pm1} = \beta_{22}^{p=\pm1} \cdot A_{22}^{k=0,(i)}$$
$$C\tilde{S}_{22,k=0,p=\pm1}^{(e)} = wc1 \cdot X_{2,k=0,p=\pm1}^{(e)} \cdot w1_{22}^{k=0,p=\pm1}$$
$$S\tilde{S}_{22,k=0,p=\pm1}^{(e)} = ws1 \cdot X_{2,k=0,p=\pm1}^{(e)} \cdot w1_{22}^{k=0,p=\pm1}$$
$$wc1X = wc1 \cdot \tilde{X}_{n=2,k=0,p=\pm1}, \qquad ws1X = ws1 \cdot \tilde{X}_{n=2,k=0,p=\pm1}$$

where

$$X_{n=2,k=0,p=\pm1} = 1.5e, \qquad \tilde{X}_{n=2,k=0,p=\pm1} = 1.5$$
$$X_{n=2,k=0,p=\pm1}^{(e)} = 1.5, \qquad (wc1 = -S_{22}, ws1 = C_{22})$$
$$C^1\tilde{S}_{22,k=0,p=\pm1}^{(L)} = \left(6 + \frac{3\gamma}{q}\tilde{\gamma}_2^{p=\pm1}\right) \cdot wc1X \cdot w1_{22}^{k=0,p=\pm1}$$
$$S^1\tilde{S}_{22,k=0,p=\pm1}^{(L)} = \left(6 + \frac{3\gamma}{q}\tilde{\gamma}_2^{p=\pm1}\right) \cdot ws1X \cdot w1_{22}^{k=0,p=\pm1}$$
$$C^1\tilde{S}_{22,k=0,p=\pm1}^{(i)} = wc1X \cdot w2_{22}^{k=0,p=\pm1}$$
$$S^1\tilde{S}_{22,k=0,p=\pm1}^{(i)} = ws1X \cdot w2_{22}^{k=0,p=\pm1}$$
$$C^1\tilde{S}_{22,k=0,p=\pm1}^{(l)} = p \cdot ws1X \cdot w1_{22}^{k=0,p=\pm1}$$
$$S^1\tilde{S}_{22,k=0,p=\pm1}^{(l)} = -p \cdot wc1X \cdot w1_{22}^{k=0,p=\pm1}$$
$$C^1\tilde{S}_{22,k=0,p=\pm1}^{(\Omega)} = 2 \cdot ws1X \cdot w1_{22}^{k=0,p=\pm1}$$
$$S^1\tilde{S}_{22,k=0,p=\pm1}^{(\Omega)} = 2 \cdot wc1X \cdot w1_{22}^{k=0,p=\pm1}$$

Appendix K: Tesseral Periodics

The AP and NA models calculate periodics from tesseral harmonics up to the eighth order. The perturbation equations are based on Kaula [4].

The BAXBOZ subroutine calculates periodic perturbations using sines and cosines of angular variables, right ascension of Greenwich at the time of prediction, and coefficients calculated by BAX1. For resonance cases, the respective perturbations are treated elsewhere [10].

To avoid excessive arithmetic calculations,

$$\begin{matrix}\cos\\ \sin\end{matrix}\psi_{qkp} = \begin{matrix}\cos\\ \sin\end{matrix}(q(\Omega - S_{\text{star}}) + k\omega + pl)$$

are calculated with the formulas

$$k = p = 0, \qquad q = 1, \ldots, 8$$
$$\cos(\Omega - S_{\text{star}}) = c1, \qquad \sin(\Omega - S_{\text{star}}) = s1$$
$$\cos 2(\Omega - S_{\text{star}}) = \cos^2(\Omega - S_{\text{star}}) - \sin^2(\Omega - S_{\text{star}})$$
$$\sin 2(\Omega - S_{\text{star}}) = 2\cos(\Omega - S_{\text{star}})\sin(\Omega - S_{\text{star}})$$
$$\cos 3(\Omega - S_{\text{star}}) = \cos 2(\Omega - S_{\text{star}})\cos(\Omega - S_{\text{star}})$$
$$- \sin 2(\Omega - S_{\text{star}})\sin(\Omega - S_{\text{star}})$$
$$\sin 3(\Omega - S_{\text{star}}) = \sin 2(\Omega - S_{\text{star}})\cos(\Omega - S_{\text{star}})$$
$$+ \cos 2(\Omega - S_{\text{star}})\sin(\Omega - S_{\text{star}})$$
$$\ldots$$
$$q = 1, \ldots, 7$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\alpha = \pm\omega, \qquad \beta = q(\Omega - S_{\text{star}})$$

For $p = +1$ and -1, we have the following functions:

$$\begin{matrix}\cos\\ \sin\end{matrix}\psi_{qkp}\colon\ qkp = 111,\ 1-1-1,\ 201,\ 20-1,\ 211,\ 2-1-1,\ 311,\ 3-1-1$$

Functions and their derivatives are calculated as follows.
For $e^i = L, i$,

$$\begin{aligned}\overset{(e_i=L,i)}{\sum} &= \sum_{q=1}^{8}[(C_\Sigma \tilde{S}^{(e_i)}_{q,k=p=0})\cdot\cos\psi_{q00} + (S_\Sigma \tilde{S}^{(e_i)}_{q,k=p=0})\cdot\sin\psi_{q00}]\\ &+ \sum_{k=p=\pm1}\sum_{q=1}^{3}[(C_\Sigma \tilde{S}^{(e_i)}_{q,k=p=\pm1})\cos\psi_{qkp}\\ &+ (S_\Sigma \tilde{S}^{(e_i)}_{q,k=p=\pm1})\sin\psi_{qkp}]\\ &+ e\Big\{\sum_{k=\pm1}\sum_{q=1}^{7}[(C^1_\Sigma \tilde{S}^{(e_i)}_{q,k=\pm1,p=0})\cos\psi_{qk0}\\ &+ (S^1_\Sigma \tilde{S}^{(e_i)}_{q,k=\pm1,p=0})\sin\psi_{qk0}]\\ &+ \sum_{p=\pm1}[(C^1\tilde{S}^{(e_i)}_{22,k=0,p=\pm1})\cos\psi_{20p} + (S^1\tilde{S}^{(e_i)}_{22,k=0,p=\pm1})\sin\psi_{20p}]\Big\}\end{aligned}$$

For $e^i = e$,

$$\begin{aligned}\overset{e}{\sum} &= e\Big\{\sum_{q=1}^{8}[(C^1_\Sigma \mathbf{S}^{(e)}_{q00})\cdot\cos\psi_{q00} + (S^1_\Sigma \mathbf{S}^{(e)}_{q00})\cdot\sin\psi_{q00}]\\ &+ \sum_{k=p=\pm1}\sum_{q=1}^{3}[(C^1_\Sigma \mathbf{S}^{(e)}_{qkp})\cdot\cos\psi_{qkp} + (S^1_\Sigma \mathbf{S}^{(e)}_{qkp})\cdot\sin\psi_{qkp}]\Big\}\\ &+ \sum_{k=\pm1}\sum_{q=1}^{7}[(C_\Sigma \mathbf{S}^{(e_i)}_{qk0})\cdot\cos\psi_{qk0} + (S_\Sigma \mathbf{S}^{(e)}_{qk0})\cdot\sin\psi_{qk0}]\\ &+ \sum_{p=\pm1}[(C\mathbf{S}^{(e_i)}_{20p})\cdot\cos\psi_{20p} + (S\mathbf{S}^{(e)}_{20p})\cdot\sin\psi_{20p}]\end{aligned}$$

For $e^i = \omega$,

$$\begin{aligned}\overset{(\omega)}{\sum} &= \sum_{k=p=\pm1}\sum_{q=1}^{3}[(C_\Sigma \mathbf{S}^{(\omega)}_{qkp})\cdot\cos\psi_{qkp} + (S_\Sigma \mathbf{S}^{(\omega)}_{qkp})\cdot\sin\psi_{qkp}]\\ &+ e\sum_{k=\pm1}\sum_{q=1}^{7}[(C^1_\Sigma \mathbf{S}^{(\omega)}_{qk0})\cdot\cos\psi_{qk0} + (S^1_\Sigma \mathbf{S}^{(\omega)}_{qk0})\cdot\sin\psi_{qk0}]\end{aligned}$$

For $e^i = \Omega$,

$$\begin{aligned}\overset{(\Omega)}{\sum} &= \sum_{q=1}^{8}[(C_\Sigma \tilde{S}^{(\Omega)}_{q,k=p=0})\cdot\cos\psi_{q00} + (S_\Sigma \tilde{S}^{(\Omega)}_{q,k=p=0})\cdot\sin\psi_{q00}]\\ &+ \sum_{k=p=\pm1}\sum_{q=1}^{3}[(C_\Sigma \tilde{S}^{(\Omega)}_{q,k=p=\pm1})\cos\psi_{qkp}\\ &+ (S_\Sigma \tilde{S}^{(\Omega)}_{q,k=p=\pm1})\sin\psi_{qkp}]\\ &+ e\Big\{\sum_{k=\pm1}\sum_{q=1}^{7}[(C^1_\Sigma \tilde{S}^{(\Omega)}_{q,k=\pm1,p=0})\cos\psi_{qk0}\\ &+ (S^1_\Sigma \tilde{S}^{(\Omega)}_{q,k=\pm1,p=0})\sin\psi_{qk0}]\\ &+ \sum_{p=\pm1}[(C^1\tilde{S}^{(\Omega)}_{22,k=0,p=\pm1})\cos\psi_{20p} + (S^1\tilde{S}^{(\Omega)}_{22,k=0,p=\pm1})\sin\psi_{20p}]\Big\}\end{aligned}$$

For $e^i = l$, there are only two sums corresponding to this case. They contain the symbols $p = \pm1$, $k = 0$, and $p = k = \pm1$. But in the second case, the differentiation with respect to l can be replaced by differentiation with respect to ω:

$$\begin{aligned}\overset{(l)}{\sum} &= e\Big\{\sum_{p=\pm1}[(C^1\tilde{S}^{(l)}_{22,k=0,p=\pm1})\cos\psi_{20p}\\ &+ (S^1\tilde{S}^{(l)}_{22,k=0,p=\pm1})\sin\psi_{20p}]\Big\}\\ &+ \sum_{k=p=\pm1}\sum_{q=1}^{3}[(C_\Sigma \tilde{S}^{(\omega)}_{q,k=p=\pm1})\cos\psi_{qkp}\\ &+ (S_\Sigma \tilde{S}^{(\omega)}_{q,k=p=\pm1})\sin\psi_{qkp}]\end{aligned}$$

Substituting these equations into the Eqs. (J1) gives the following result:

$$\begin{cases}\Delta L = L\Sigma^{(l)}\\ \Delta i = \dfrac{\gamma}{\eta}(\theta\Sigma^{(\omega)} - \Sigma^{(\Omega)})\\ \Delta\Omega = \dfrac{\gamma s_1}{\eta}\Sigma^{(i)} & \gamma = \dfrac{n}{n_e}\\ e\Delta\omega = \gamma\eta\Sigma^{(e)} - e\theta\Delta\Omega\\ \Delta\lambda = \gamma\left(\Sigma^{(L)} + \dfrac{e\eta}{1+\eta}\Sigma^{(e)}\right) - \theta\Delta\Omega\end{cases}$$

Finally, the singularity in the equation for Δe, which emerges because of division by e, is resolved as follows:

$$\overset{(\omega)}{\sum} = \overset{(\omega)}{\sum_1} + e\overset{(\omega)}{\sum_2},\qquad \overset{(l)}{\sum} = \overset{(l)}{\sum_1} + e\overset{(l)}{\sum_2} = \overset{(\omega)}{\sum_1} + e\overset{(l)}{\sum_2}$$

$$\overset{(\omega)}{\sum_1} = \sum_{k=p=\pm1}\sum_{q=1}^{3}[(C_\Sigma \mathbf{S}^{(e_i)}_{qkk}\cos\psi_{qkk} + (S_\Sigma \mathbf{S}^{(e_i)}_{qkk})\sin\psi_{qkk}];$$

$$\overset{(\omega)}{\sum_2} = \sum_{k=\pm1}\sum_{q=1}^{7}[(C^1_\Sigma \mathbf{S}^{(\omega)}_{qk0})\cdot\cos\psi_{qk0} + (S^1_\Sigma \mathbf{S}^{(\omega)}_{qk0})\cdot\sin\psi_{qk0}],$$

$$\overset{(l)}{\sum_2} = \sum_{p=\pm1}[(C^1\tilde{S}^{(l)}_{22,k=0,p=\pm1})\cos\psi_{20p} + (S^1\tilde{S}^{(l)}_{22,k=0,p=\pm1})\sin\psi_{20p}];$$

$$\begin{aligned}\Delta e &= \gamma\eta(\eta\Sigma^{(l)}_2 - \Sigma^{(\omega)}_2) + \gamma\eta\frac{(\eta-1)}{e}\Sigma^{(\omega)}_1\\ &= \gamma\eta(\eta\Sigma^{(l)}_2 - \Sigma^{(\omega)}_2) - \frac{\gamma\eta e}{1+\eta}\Sigma^{(\omega)}_1\end{aligned}$$

Appendix L: Position and Velocity

Position and velocity are calculated in the following sequence:

$$s_n = s'' + \theta''\Delta i,\qquad \eta_n = \eta'' - e''\frac{\Delta e}{\eta''},\qquad a_n = \frac{L_n^2}{\mu}$$

$$\varepsilon'' = E'' + g'',\qquad \Delta\varepsilon = \frac{\Delta\lambda + \Delta k\sin\varepsilon'' - \Delta h\cos\varepsilon''}{1 - h''\sin\varepsilon'' - k''\cos\varepsilon''}$$

$$\varepsilon_n = \varepsilon'' + \Delta\varepsilon,\qquad \tilde{x} = a_n(\cos\varepsilon_n - k_n + bh_n)$$

$$\tilde{y} = a_n(\sin\varepsilon_n - h_n - bk_n),\qquad \dot{\tilde{x}} = -\frac{\mu}{L_n\eta_n}\left(h_n + \frac{\tilde{y}}{r}\right)$$

$$\dot{\tilde{y}} = \frac{\mu}{L_n\eta_n}\left(k_n + \frac{\tilde{x}}{r}\right)$$

where

$$r = a_n(1 - k_n \cos\varepsilon_n - h_n \sin\varepsilon_n), \qquad b = \frac{k_n \sin\varepsilon_n - h_n \cos\varepsilon_n}{1 + \eta_n}$$

$$\bar{P} = \begin{pmatrix} P_1 = \cos\Omega_n \\ P_2 = \sin\Omega_n \\ P_3 = 0 \end{pmatrix}, \qquad \bar{Q} = \begin{pmatrix} Q_1 = -\sin\Omega_n\theta_n \\ Q_2 = \cos\Omega_n\theta_n \\ Q_3 = s_n \end{pmatrix}$$

$$\bar{r} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \bar{P}\,\tilde{x} + \bar{Q}\,\tilde{y}, \qquad \dot{\bar{r}} = \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \bar{P}\,\dot{\tilde{x}} + \bar{Q}\,\dot{\tilde{y}}$$

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